Sur le nombre maximal de façons de paver le plan par translation

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École des jeunes chercheurs
GDR-IM
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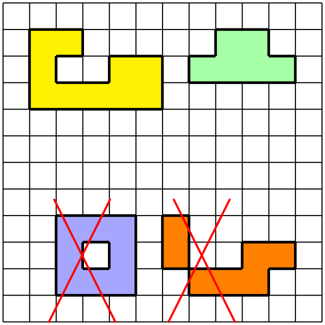
Travail commun avec : Alexandre Blondin Massé, Srečko Brlek et Ariane Garon

Plan

1. Introduction
   - Figures discrètes
   - Pavages
   - Beauquier et Nivat
   - Tuiles hexagonales et tuiles carrées
   - Une conjecture de Brlek, Dulucq, Fédou et Provençal
2. Preuve de la conjecture
3. Conclusion

Discrete Figures and Polyominoes

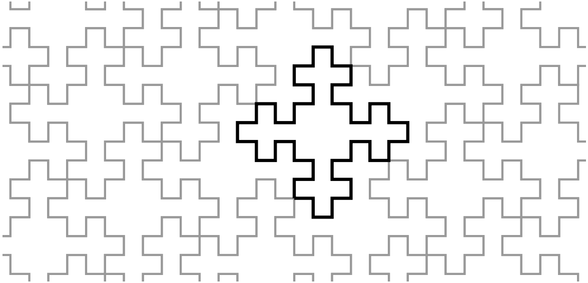
- Discrete plane : $\mathbb{Z}^2$
- **Definition** : A polyomino is a finite, 4-connected subset of the plane, without holes.



The Tiling by Translation Problem

Let P be a polyomino. We say that

- P tiles the plane if there exists a set T of non-overlapping translated copies of P that covers all the plane.
- P is called a tile if it tiles the plane.



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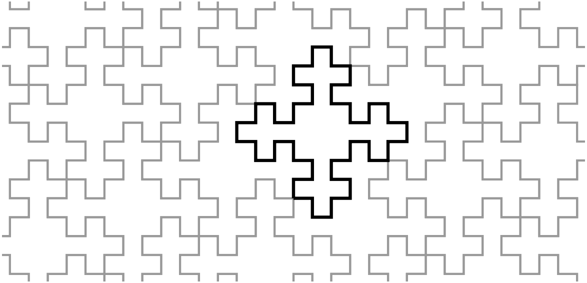
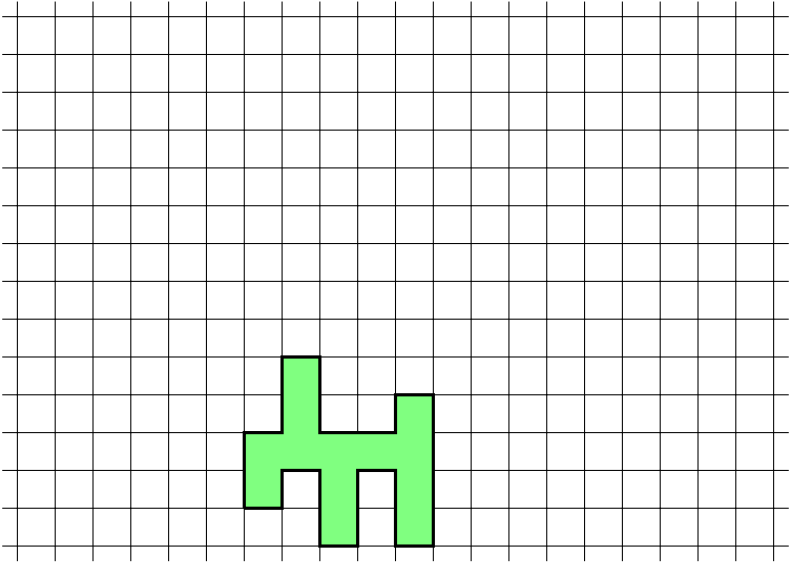
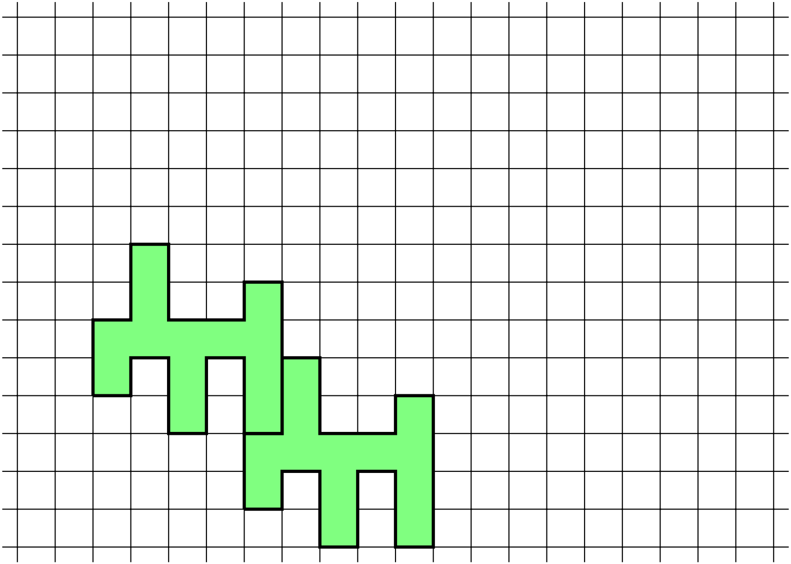
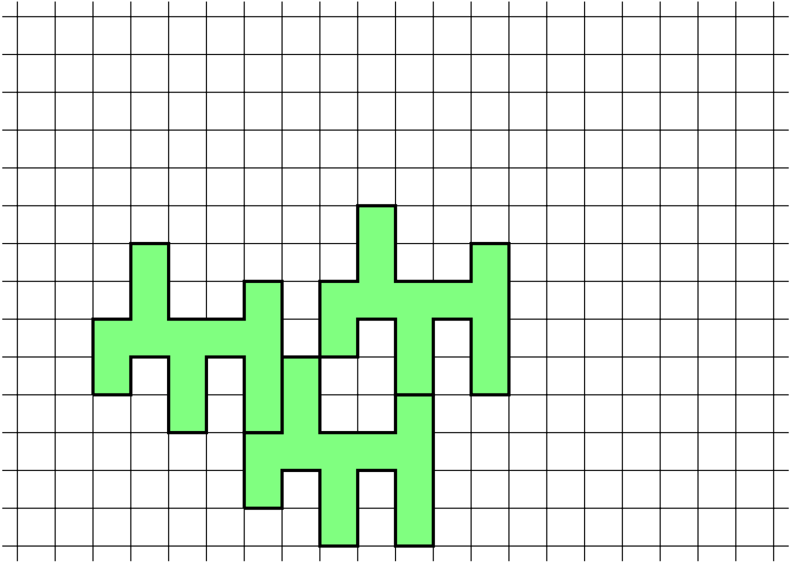


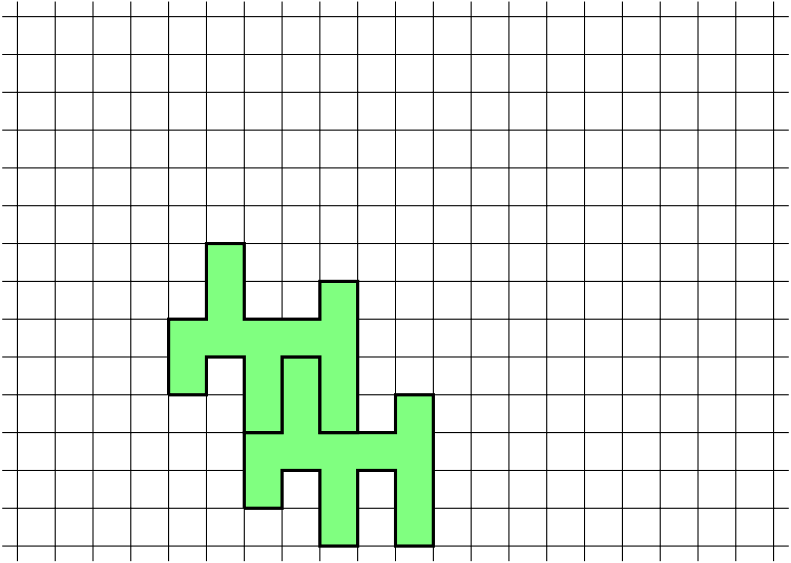
Problem

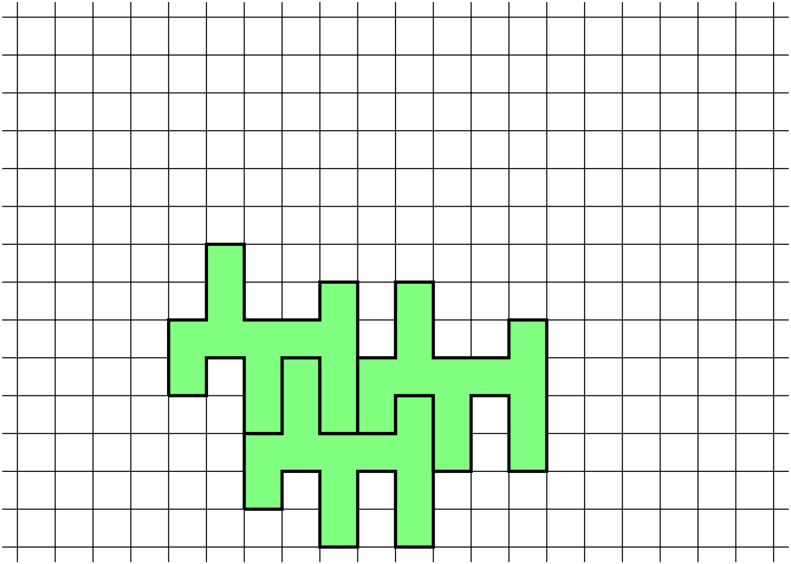
Does a given polyomino P tile the plane?

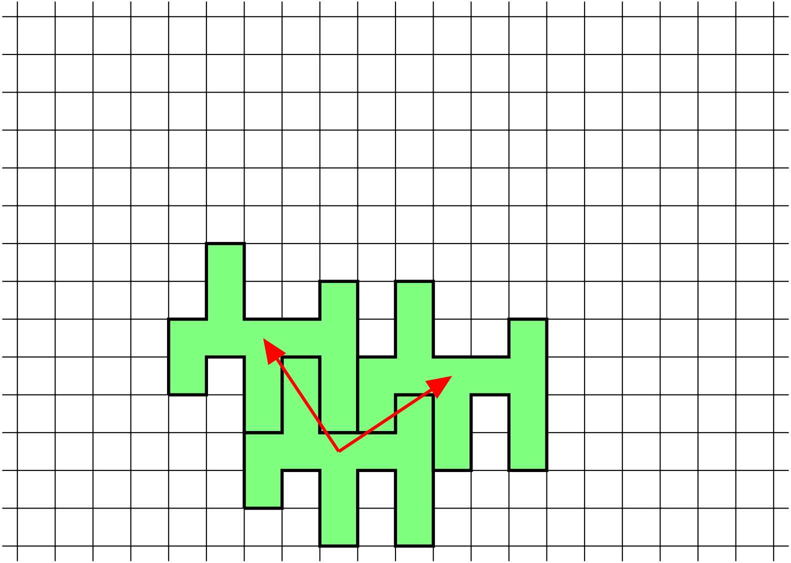


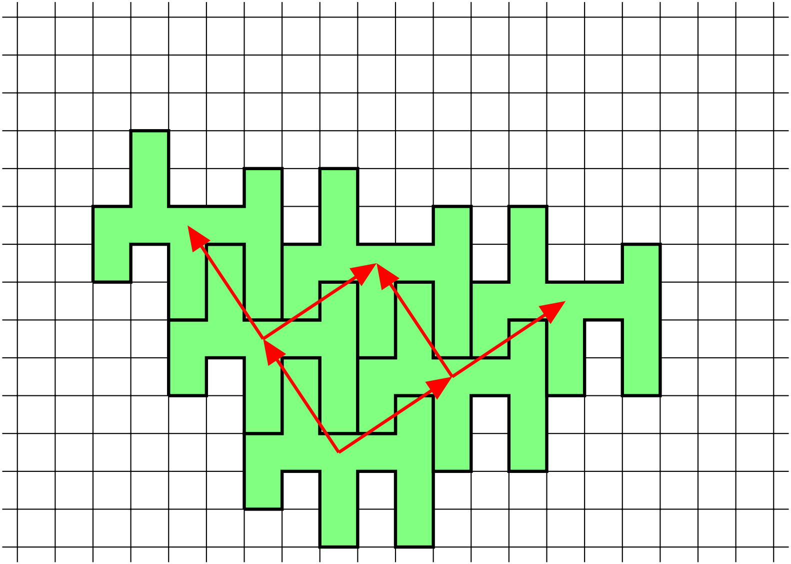


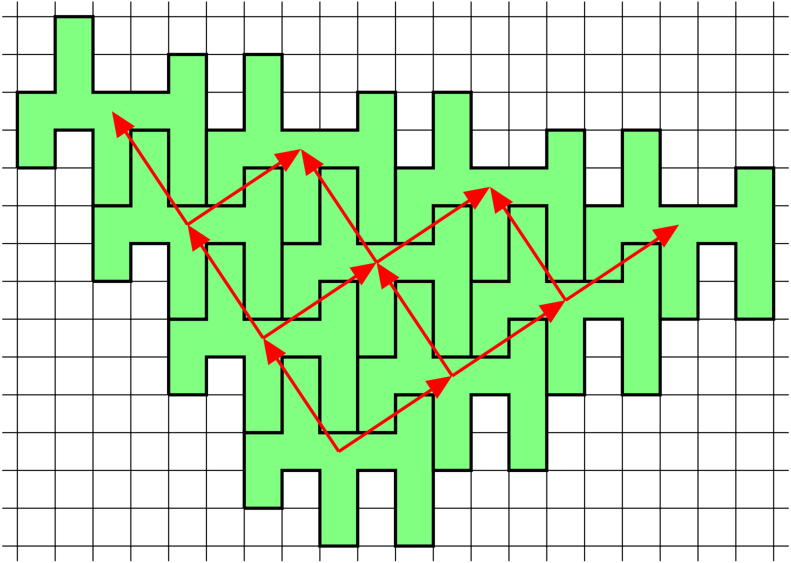


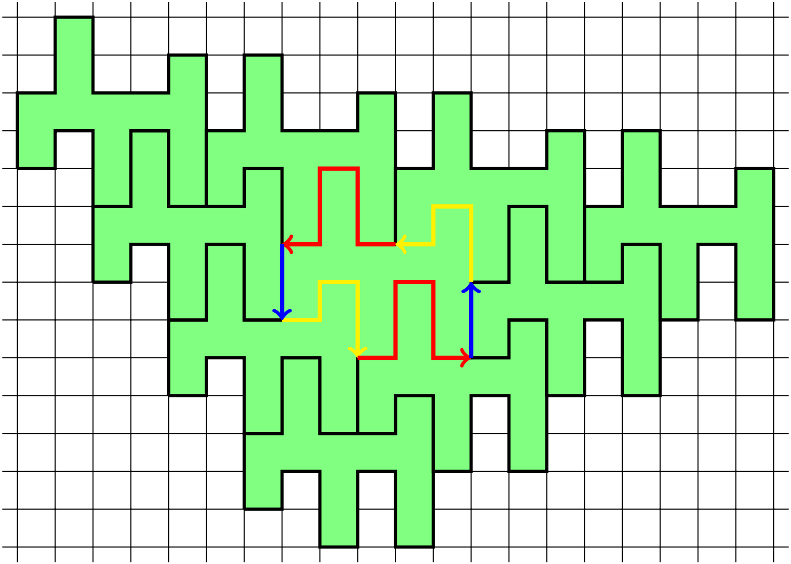








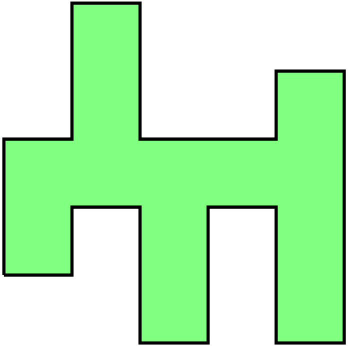




Freeman Chain Code

$$\Sigma = \mathbb{Z}_4 = \{\mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3}\}$$

$\mathbf{0} \rightarrow$	$\mathbf{1} \uparrow$
$\mathbf{2} \leftarrow$	$\mathbf{3} \downarrow$



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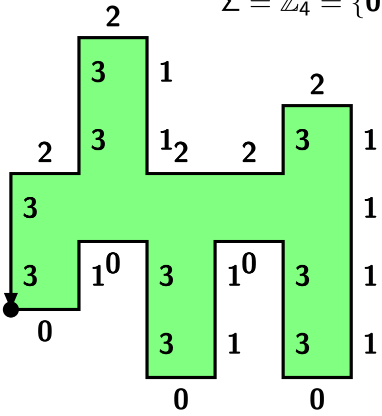

$w =$ **01033011033011112322211233233**

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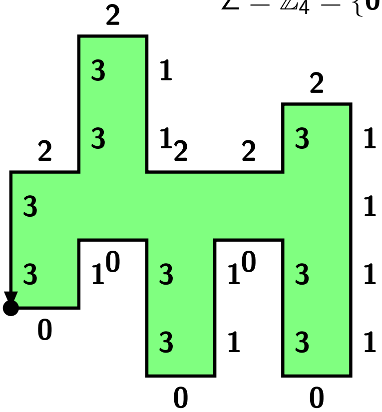

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Any conjugate w' of w codes the same polyomino.

w and w' are conjugate if there exist $u, v \in \Sigma^*$ such that $w = uv$ and $w' = vu$. We write $w \equiv_{|u|} w'$.

$w = \mathbf{0103301103301111232211233233}$

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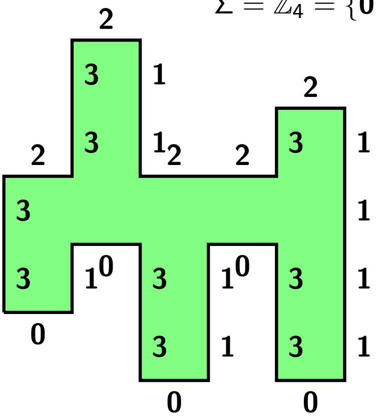

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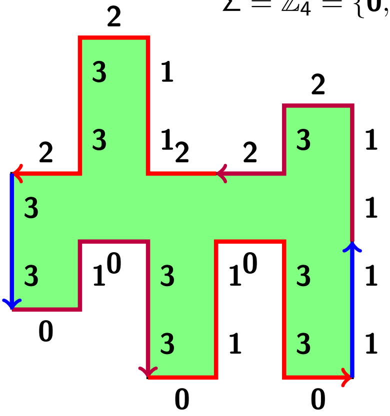

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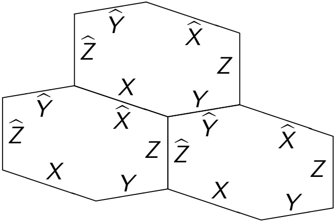
Characterization : A polyomino P tiles the plane if and only if there exist $X, Y, Z \in \Sigma^*$ such that $w \equiv XYZ\widehat{X}\widehat{Y}\widehat{Z}$.

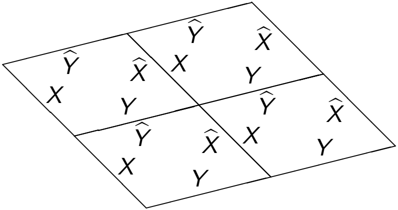
$X = \mathbf{0\ 0\ 1\ 0\ 3\ 0\ 1}$ 

$\widehat{X} = \mathbf{3\ 2\ 1\ 2\ 3\ 2\ 2}$ 

hexagon tiles



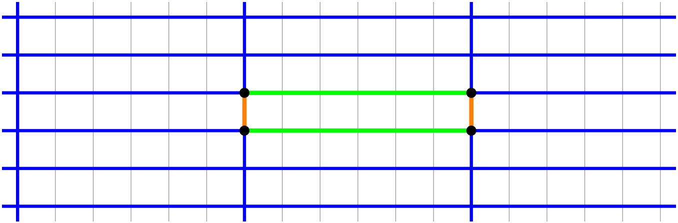
square tiles



Maurits Cornelis Escher (1898-1972). Hexagonal tiling. Square tiling.

Hexagonal Tilings

There are polyominoes admitting many hexagon tilings :



A $1 \times n$ rectangle tiles the plane as an hexagon in $n-1$ ways and as a square in only 1 way.

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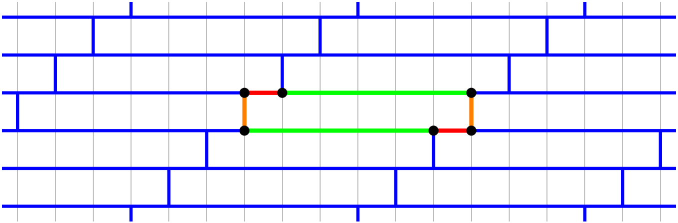

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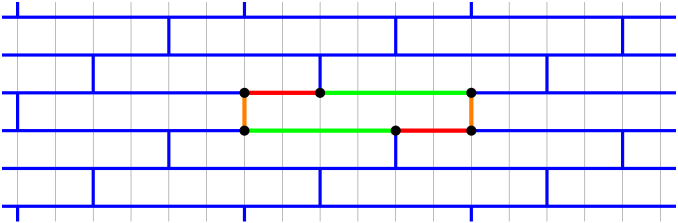

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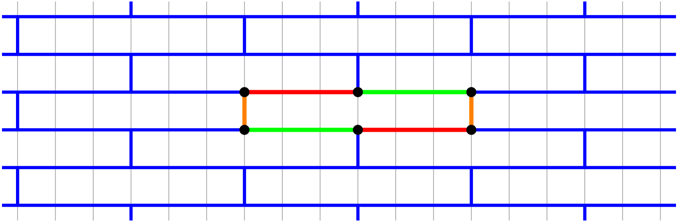

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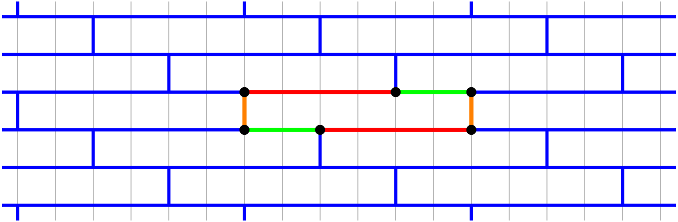

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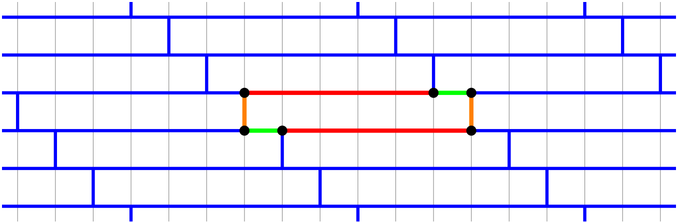

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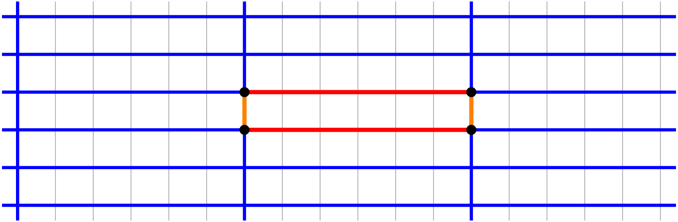

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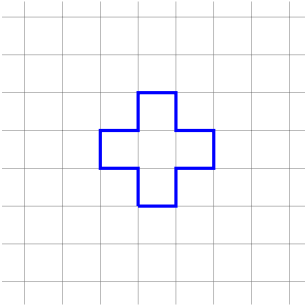
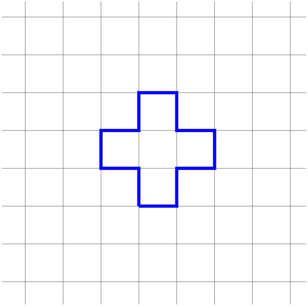
Square Tilings

The pentamino has two distinct square factorizations :





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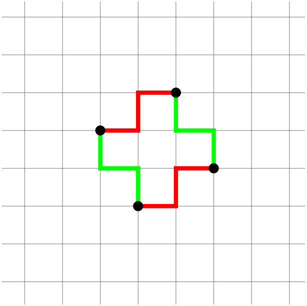

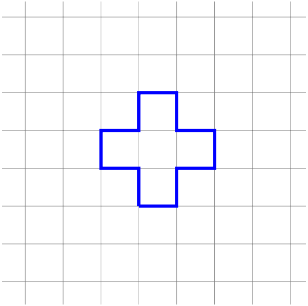

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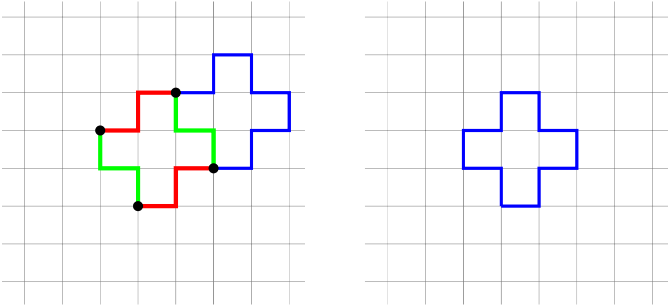

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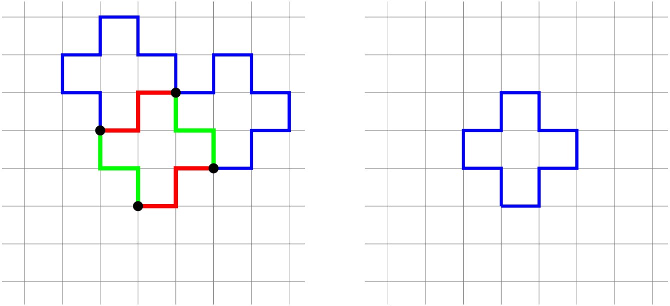

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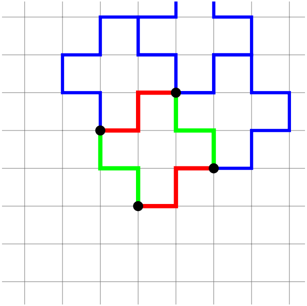

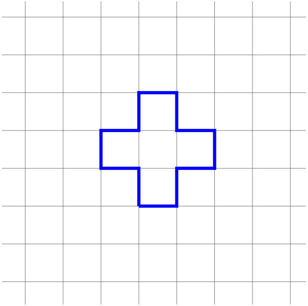

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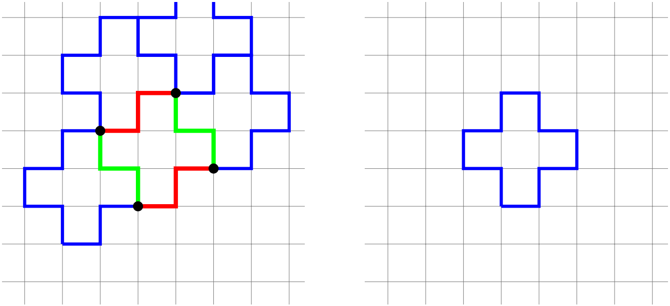

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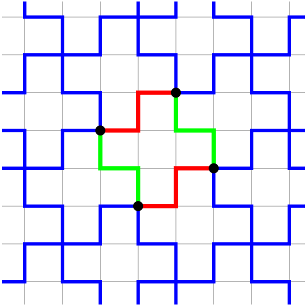

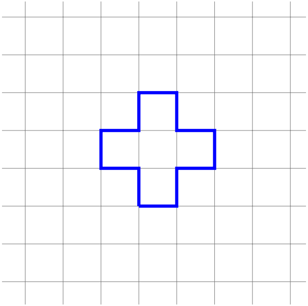

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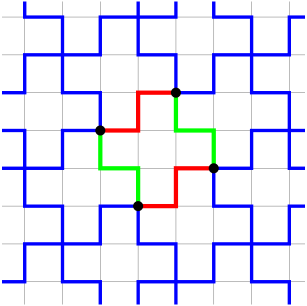

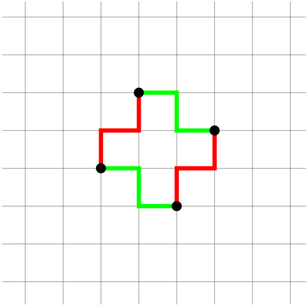

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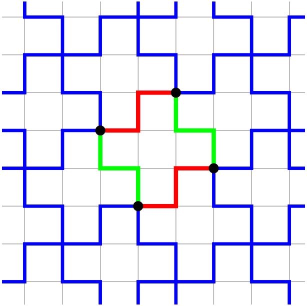

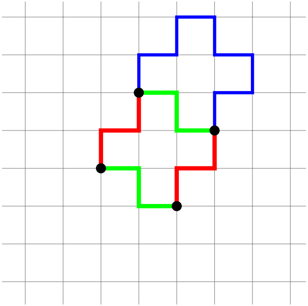

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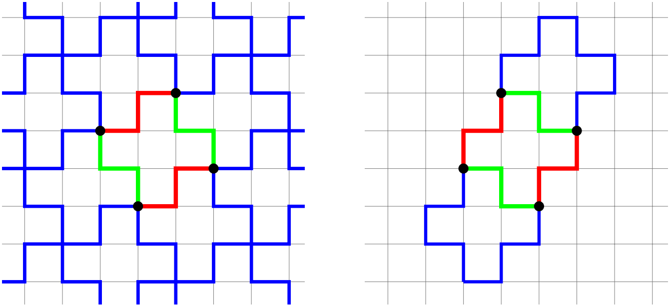

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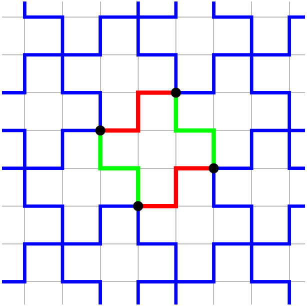

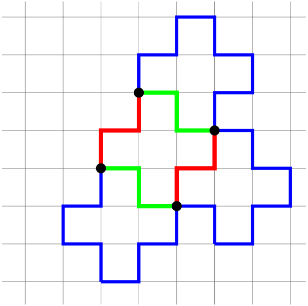

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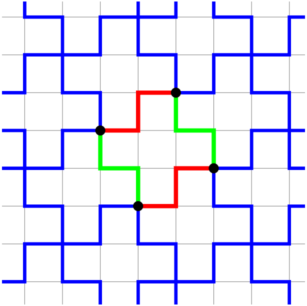

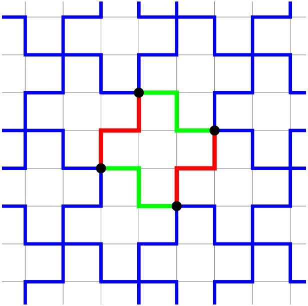

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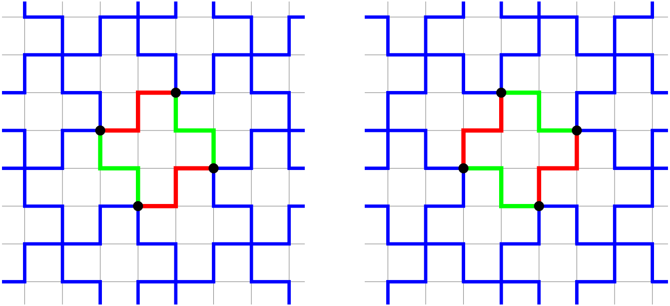


Conjecture (Brlek, Dulucq, Fédou, Provençal 2007)

A tile has at most 2 square factorizations.

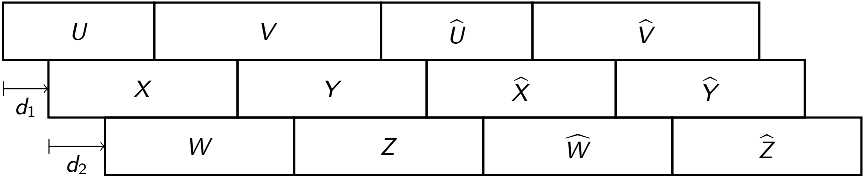
Preuve

Supposons qu'il existe une tuile triple carrée dont la frontière s'écrit :

$$UV\widehat{U}\widehat{V} \equiv_{d_1} XY\widehat{X}\widehat{Y} \equiv_{d_2} WZ\widehat{W}\widehat{Z}.$$

Lemma (Brlek, Fédou, Provençal, 2008)

Les deux factorisations $UV\widehat{U}\widehat{V} \equiv_{d_1} XY\widehat{X}\widehat{Y}$ *d'une tuile double carrée doivent alterner c'est-à-dire que* $0 < d_1 < |U| < d_1 + |X|$.

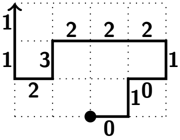


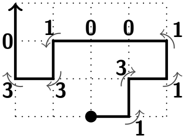
Suite des différences successives

La suite des différences successives de $w \in \Sigma^*$ est

$$\Delta w = (w_2 - w_1) \cdot (w_3 - w_2) \cdots (w_n - w_{n-1}).$$

Elle représente la suite des virages d'un chemin.





$w = \mathbf{01012223211}$ $\Delta w = \mathbf{1311001330}$

On considère aussi $\overset{\circ}{\Delta} w$ bien définie sur les classes de conjugaison :

$$\overset{\circ}{\Delta} w = (w_2 - w_1) \cdot (w_3 - w_2) \cdots (w_n - w_{n-1}) \cdot (w_1 - w_n) = \Delta w \cdot (w_1 - w_n).$$

Turning number

Le turning number d'un chemin fermé w est

$$\mathcal{T}(\mathring{\Delta}\, w) = \frac{|\mathring{\Delta}\, w|_{\mathbf{1}} - |\mathring{\Delta}\, w|_{\mathbf{3}}}{4}$$

et correspond à sa courbure totale divisée par 2π. On a

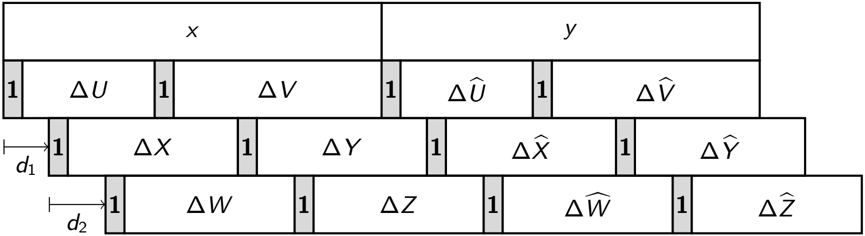
- $\mathcal{T}(\Delta w) = -\mathcal{T}(\Delta \widehat{w})$ pour tout chemin $w \in \Sigma^*$
- $\mathcal{T}(\mathring{\Delta}\, w) = \pm 1$ pour tout chemin simple et fermé w.

Lemma

Si $XY\widehat{X}\widehat{Y}$ est la frontière orientée positivement d'une tuile carrée, alors

$$\mathring{\Delta}\, XY\widehat{X}\widehat{Y} = \Delta X \cdot \mathbf{1} \cdot \Delta Y \cdot \mathbf{1} \cdot \Delta \widehat{X} \cdot \mathbf{1} \cdot \Delta \widehat{Y} \cdot \mathbf{1}.$$

Preuve



On s'intéresse aux moitiés du contour

$$
\begin{aligned}
x &= x_0 x_1 x_2 \cdots x_{n-1} = \mathbf{1} \cdot \Delta U \cdot \mathbf{1} \cdot \Delta V, \\
y &= y_0 y_1 y_2 \cdots y_{n-1} = \mathbf{1} \cdot \Delta \widehat{U} \cdot \mathbf{1} \cdot \Delta \widehat{V}.
\end{aligned}
$$

Preuve

On définit trois réflexions sur $\mathbb{Z}_n$:

$$\begin{aligned} s_1 : i &\mapsto (|U| - i) \mod n, \\ s_2 : i &\mapsto (|X| + 2d_1 - i) \mod n, \\ s_3 : i &\mapsto (|W| + 2(d_1 + d_2) - i) \mod n. \end{aligned}$$

Lemma

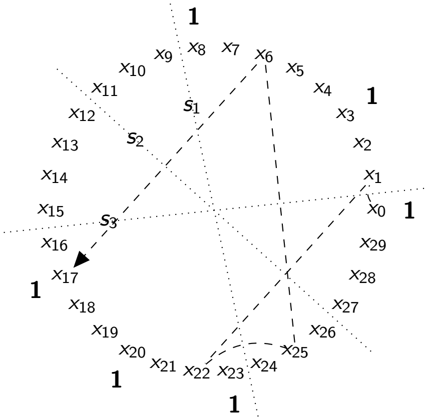
Soit $i \in \mathbb{Z}_n$ et $j \in \{1, 2, 3\}$ tels que s_j est admissible sur i. Alors

- $y_i = \overline{x_{s_j(i)}}$ *et* $x_i = \overline{y_{s_j(i)}}$.

où $\overline{\mathbf{0}} = \mathbf{0}$, $\overline{\mathbf{1}} = \mathbf{3}$, $\overline{\mathbf{2}} = \mathbf{2}$ *et* $\overline{\mathbf{3}} = \mathbf{1}$.

Preuve

Soit $n = 30$, $d_1 = 3$, $d_2 = 5$, $|U| = 17$, $|X| = 17$ et $|W| = 15$.



On a $s_1 = s_3 s_2 s_1 s_3 s_2$. Si $s_3 s_2 s_1 s_3 s_2$ est un produit admissible de réflexions sur 0, alors $x_0 = \overline{x_{17}}$ ce qui est une contradiction. Autrement, des contradictions similaires sont obtenues.

Théorème

Theorem (Blondin-Massé, Brlek, Garon, L.)

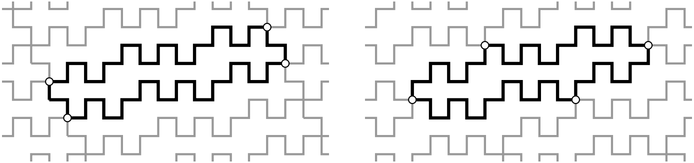
Un polyomino pave le plan à la manière d'un carré en au plus 2 façons.

Prime Double Square Tiles

Conjecture (X. Provençal and L. Vuillon, 2008)

If $XY\widehat{X}\widehat{Y}$ describes the contour of a prime double square tile, then both X and Y are palindromes.

Note : a palindrome is a word that reads the same forward as it does backward.



Crédits

- This research was driven by computer exploration using the open-source mathematical software `Sage` and its algebraic combinatorics features developed by the `Sage-Combinat` community, and in particular, F. Saliola, A. Bergeron and S. Labbé.
- Les images de ce document ont été produites à l'aide de `pgf/tikz`.

Liens :

- `sagemath.org`
- `combinat.sagemath.org`