Complexité palindromique des codages de rotations et conjectures

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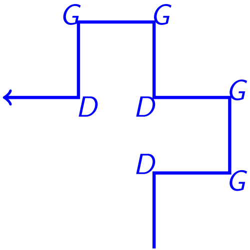
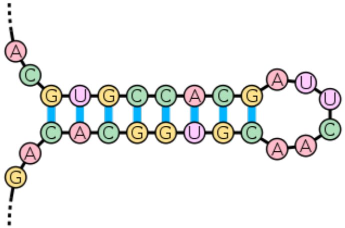
Dynamique, Arithmétique, Combinatoire
Institut de Mathématique de Luminy
Marseille, 20 avril 2010

Plan

Palindromes





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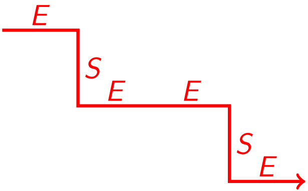

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The Fibonacci word

We define $f_{-1} = b$, $f_0 = a$ and, for $n \geq 1$,

$$f_n = f_{n-1} f_{n-2}.$$

Therefore, we have

$$\begin{array}{rcl} f_0 & = & a \\ f_1 & = & ab \\ f_2 & = & aba \\ f_3 & = & abaab \\ f_4 & = & abaababa \\ f_5 & = & abaababaabaab \\ \vdots & & \vdots \end{array}$$

The infinite word f_∞ is called the Fibonacci word.

Palindromes in the Fibonacci word

$$\mathrm{w} = a$$

Palindromes a

Palindromes in the Fibonacci word

$$\mathrm{w} = a \quad b$$

Palindromes a

b

Palindromes in the Fibonacci word

$$w = a \quad b \quad a$$

Palindromes

a

b

$a \quad b \quad a$

Palindromes in the Fibonacci word

$$w = a \quad b \quad a \quad a$$

Palindromes	a				
		b			
	a	b	a		
			a	a	

Palindromes in the Fibonacci word

$$w = a\ b\ a\ a\ b$$

Palindromes					
	a				
		b			
	a	b	a		
			a	a	
		b	a	a	b

Palindromes in the Fibonacci word

$$w = a \quad b \quad a \quad a \quad b \quad a$$

Palindromes	a					
		b				
	a	b	a			
			a	a		
		b	a	a	b	
	a	b	a	a	b	a

Palindromes in the Fibonacci word

$$w = a \quad b \quad a \quad a \quad b \quad a \quad b$$

Palindromes	a						
		b					
	a	b	a				
			a	a			
		b	a	a	b		
	a	b	a	a	b	a	
					b	a	b

Palindromes in the Fibonacci word

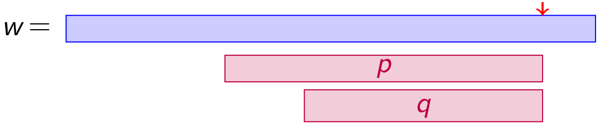
w	=	a	b	a	a	b	a	b	a

Palindromes	a							
		b						
	a	b	a					
			a	a				
		b	a	a	b			
	a	b	a	a	b	a		
					b	a	b	
				a	b	a	b	a

Palindromes in the Fibonacci word

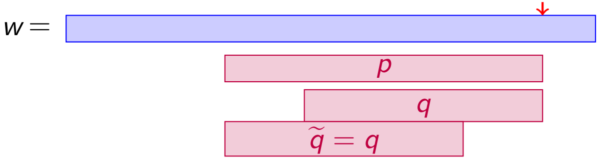
w	=	a	b	a	a	b	a	b	a	a	$\cdots$
Palindromes		a									
			b								
		a	b	a							
				a	a						
			b	a	a	b					
		a	b	a	a	b	a				
						b	a	b			
					a	b	a	b	a		
				a	a	b	a	b	a	a	
											$\ddots$

Number of distinct palindrome factors



- Assume that the first occurrence of some distinct palindromes p and q ends at the same position.

Number of distinct palindrome factors



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- Then q has a previous occurence in w which is a contradiction.

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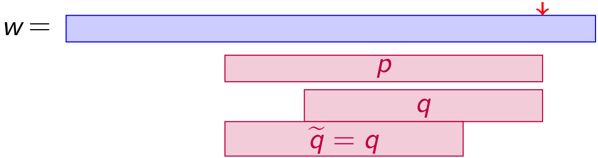


- Assume that the first occurrence of some distinct palindromes p and q ends at the same position.
- Then q has a previous occurence in w which is a contradiction.

Theorem (Droubay, Justin and Pirillo, 2001)

Let w be a finite word. Then $|\mathrm{Pal}(w)| \leq |w| + 1$.

More on the palindrome complexity

Theorem (Droubay, Justin and Pirillo, 2001)

Sturmian words w realize the upper bound, i.e. $|\mathrm{Pal}(w)| = |w| + 1$.

Definition (Brlek, Hamel, Nivat, Reutenauer, 2004)

A word w is called full if $|\mathrm{Pal}(w)| = |w| + 1$.

Since 2008, some authors say rich instead of full. Personally, I prefer to use the word full because it suggests that a limit is reached.

More on the palindrome complexity

Theorem (Brlek, Hamel, Nivat, Reutenauer, 2004)

A infinite periodic word $\mathbf{w}$ *is full if and only if*

- $\mathbf{w} = (uv)^\omega$ *with both* u *and* v *are palindromes and*
- $(uv)^{1+e}$ *is full where* $e = \lfloor \frac{||u|-|v||}{3} \rfloor / |uv|$.

Theorem (Bucci, De Luca, Glen, Zamboni, 2008)

An infinite word $\mathbf{w}$ *whose set of factors is closed under reversal is full if and only if*

$$|\mathrm{Pal}_n(\mathbf{w})| + |\mathrm{Pal}_{n+1}(\mathbf{w})| = |\mathrm{Fact}_{n+1}(\mathbf{w})| - |\mathrm{Fact}_n(\mathbf{w})| + 2$$

for all n.

What about words that are not full ?

The Thue-Morse word

We define $t_0 = a$ and, for $n \geq 1$,

$$t_n = t_{n-1}\overline{t_{n-1}}.$$

so that

$$\begin{aligned}
t_0 &= a \\
t_1 &= ab \\
t_2 &= abba \\
t_3 &= abbabaab \\
t_4 &= abbabaabbaababba \\
t_5 &= abbabaabbaababbabaababbaabbabaab \\
&\vdots
\end{aligned}$$

The infinite word t_∞ is called the Thue-Morse word.

Palindromes in the Thue-Morse

$$w = a$$

Palindromes $\quad a$

Palindromes in the Thue-Morse

$$w = a \quad b$$

Palindromes a

b

Palindromes in the Thue-Morse

$$w = a \quad b \quad b$$

Palindromes a

b

$b \quad b$

Palindromes in the Thue-Morse

$$w = a \quad b \quad b \quad a$$

Palindromes	a			
		b		
		b	b	
	a	b	b	a

Palindromes in the Thue-Morse

$$w = a \quad b \quad b \quad a \quad b$$

Palindromes					
	a				
		b			
		b	b		
	a	b	b	a	
			b	a	b

Palindromes in the Thue-Morse

w	=	a	b	b	a	b	a

Palindromes	a					
		b				
		b	b			
	a	b	b	a		
			b	a	b	
				a	b	a

Palindromes in the Thue-Morse

$$\mathrm{w} = a \quad b \quad b \quad a \quad b \quad a \quad a$$

Palindromes	a						
		b					
		b	b				
	a	b	b	a			
			b	a	b		
				a	b	a	
						a	a

Palindromes in the Thue-Morse

$$w = a \quad b \quad b \quad a \quad \color{red}{b} \quad \color{red}{a} \quad \color{red}{a} \quad \color{red}{b}$$

Palindromes								
	a							
		b						
		b	b					
	a	b	b	a				
			b	a	b			
				a	b	a		
						a	a	
					$\color{red}{b}$	$\color{red}{a}$	$\color{red}{a}$	$\color{red}{b}$

Palindromes in the Thue-Morse

w	=	a	b	b	a	b	a	a	b	b	$\cdots$
Palindromes		a									
			b								
			b	b							
		a	b	b	a						
				b	a	b					
					a	b	a				
							a	a			
						b	a	a	b		
										—	
											$\ddots$

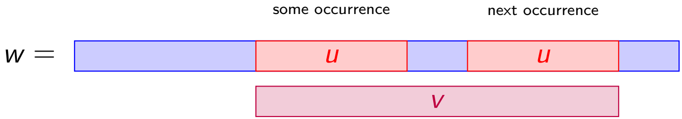
There is no new palindrome at this position : a lacuna !

Palindromes in the Thue-Morse

w	=	a	b	b	a	b	a	a	b	b	$\cdots$
Palindromes		a									
			b								
			b	b							
		a	b	b	a						
				b	a	b					
					a	b	a				
							a	a			
						b	a	a	b		
										—	
											$\ddots$

There is no new palindrome at this position : a lacuna !
Hence, the Thue-Morse word is rich in palindromes but is not full.

Complete return words



We say that v is a complete return word of u in w, if v starts at an occurrence of u and ends at the end of the next occurrence of u.

Fact

A word w is full if and only if every complete return word of a palindrome factor of w is a palindrome.

Theorem (Blondin Massé, Brlek, Garon, L., 2008)

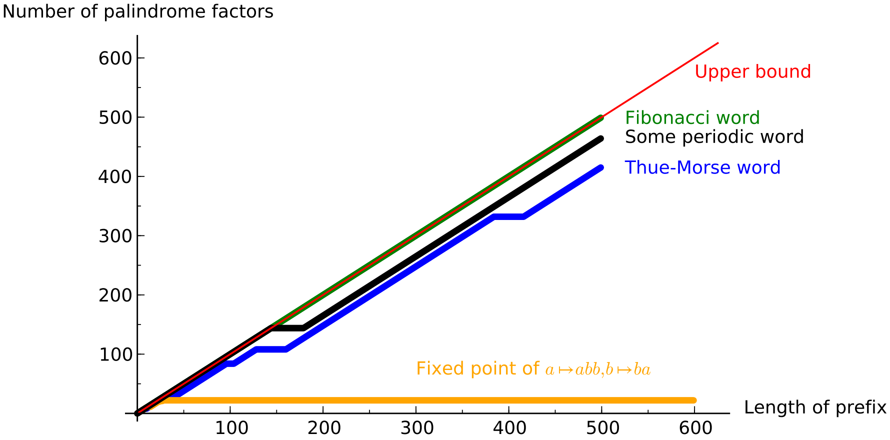
For $n \geq 1$, let $L(n)$ be the index where the n-th interval of lacunas start in the Thue-Morse word and $\ell(n)$ be its length. Then

$$L(n) = \begin{cases} 2^{n+2}, & \textit{if } n \textit{ is odd,} \\ 2^{n+2} + 2^{n+1}, & \textit{if } n \textit{ is even.} \end{cases}$$

and

$$\ell(n) = \begin{cases} 2^{n}, & \textit{if } n \textit{ is odd,} \\ 2^{n-1}, & \textit{if } n \textit{ is even.} \end{cases}$$

Palindrome complexity



Partie 2

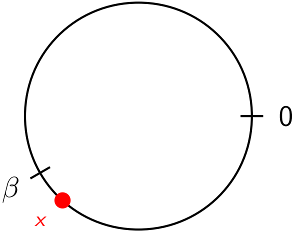
Les codages de rotation sur deux intervalles sont pleins

travail commun avec Blondin Massé, Brlek et Vuillon

Codings of rotations

The coding of rotations of parameters (x, α, β) is the word $\mathbf{C} = c_0 c_1 c_2 \cdots$ such that

$$c_i = \begin{cases} 0 & \text{if } x + i\alpha \in [0, \beta) \\ 1 & \text{if } x + i\alpha \in [\beta, 1) \end{cases}$$



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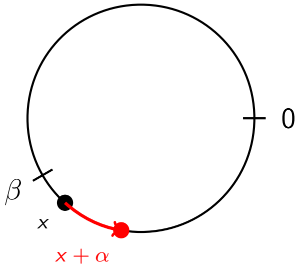

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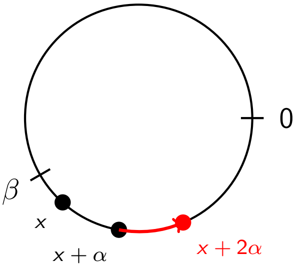

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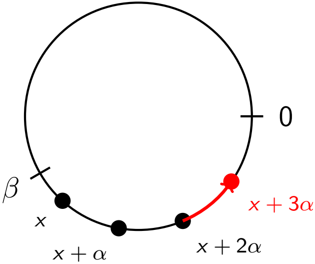

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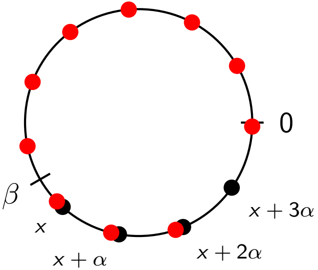


$1111000000111 \cdots$

The different cases

Let **C** be a coding of rotations of parameters (x, α, β).

- If α is rational, then **C** is periodic.
- If $\beta = 1 - \alpha$ is irrational, then **C** is Sturmian

$$|\mathrm{Fact}_n(\mathbf{C})| = n + 1.$$

- If α and β are rationally dependent, then **C** is quasi-Sturmian.

$$|\mathrm{Fact}_n(\mathbf{C})| = n + k, \quad \text{for some constant } k.$$

- Otherwise, **C** is a Rote sequence

$$|\mathrm{Fact}_n(\mathbf{C})| = 2n, \quad \text{for large enough } n.$$

Main result

Theorem (Blondin Massé, Brlek, L., Vuillon, 2009)

Every coding of rotations on two intervals is full.

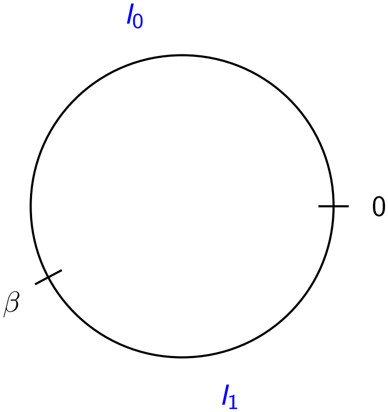
The proof is based on the following ideas :

1. Return words
2. Interval exchange transformations
3. First return function
4. Many combinatorial results on those dynamical systems

Idea of the proof

Let $x = 0.102$, $\alpha = 0.135$ and $\beta = 0.578$. Then

$$\mathbf{C} = 0000111000011110000111000011100000111000\cdots$$



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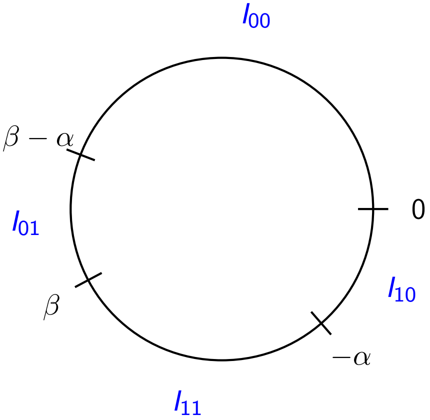

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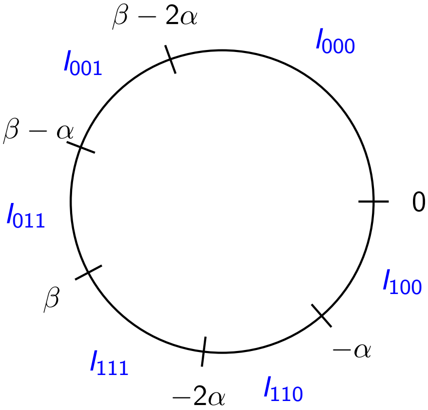

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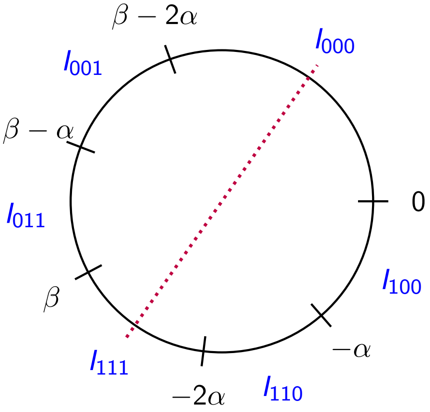

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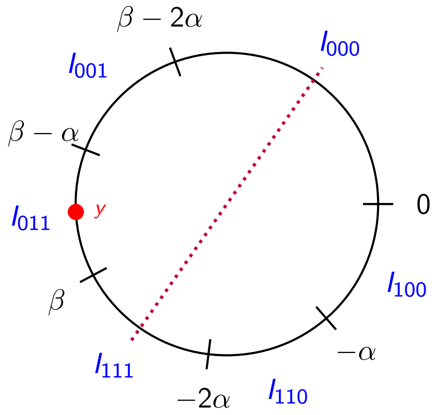

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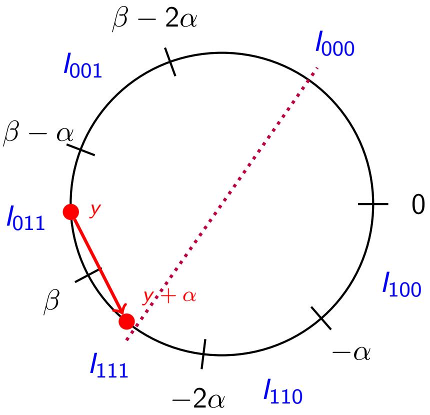

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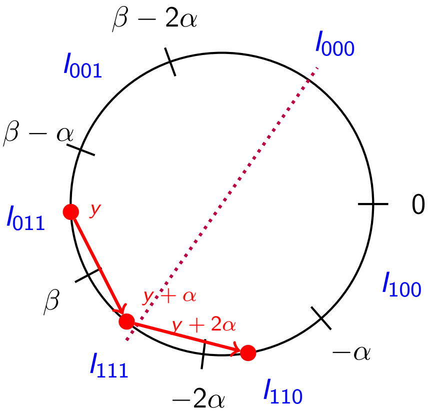

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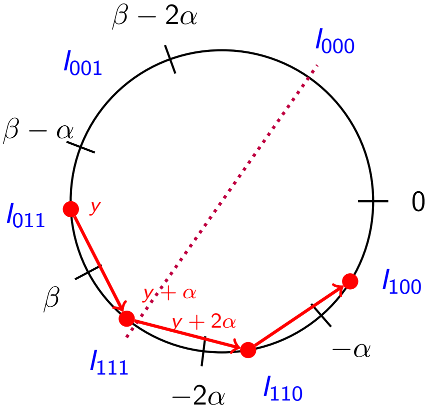

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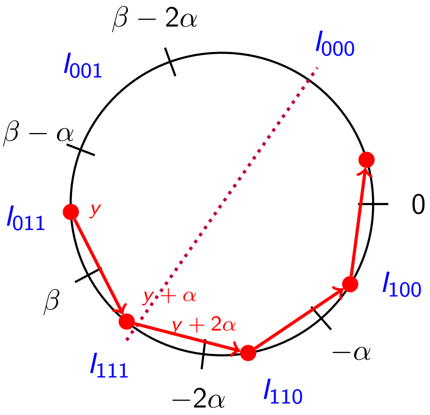

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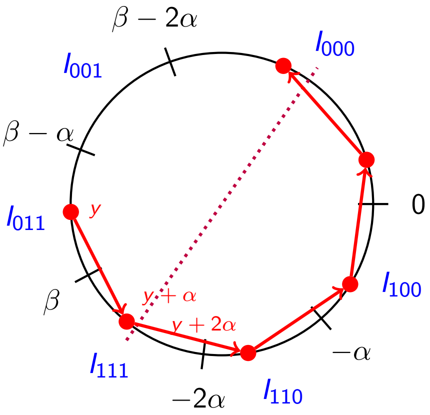

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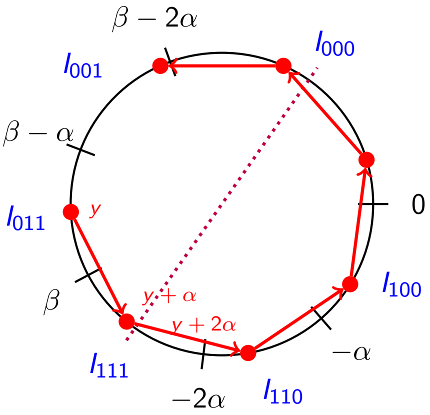

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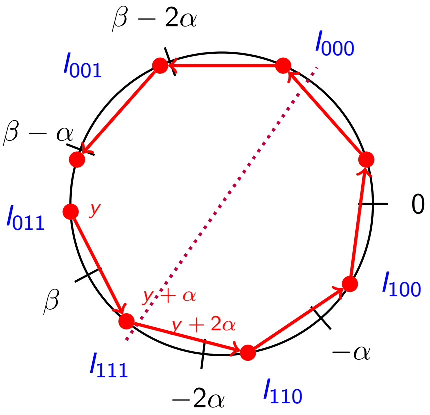

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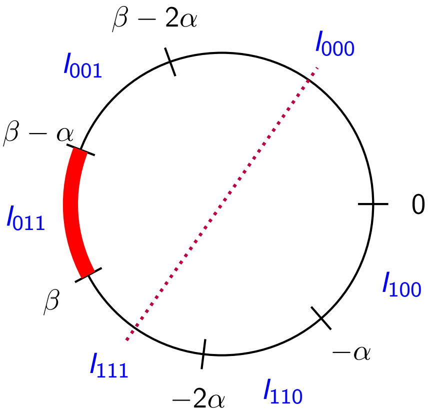

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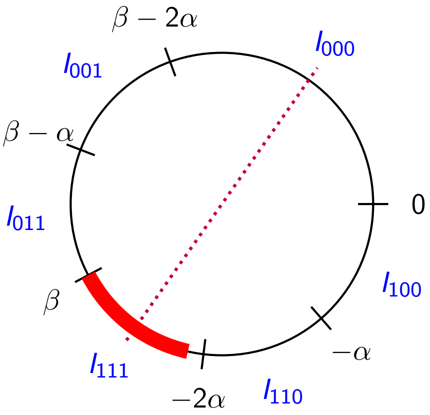

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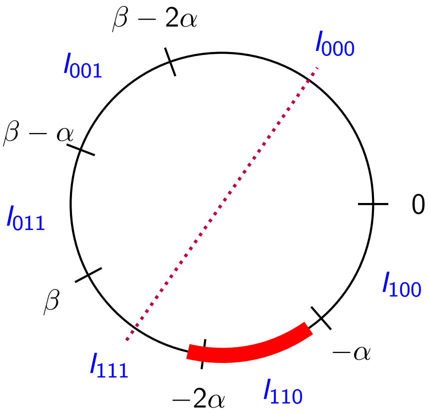

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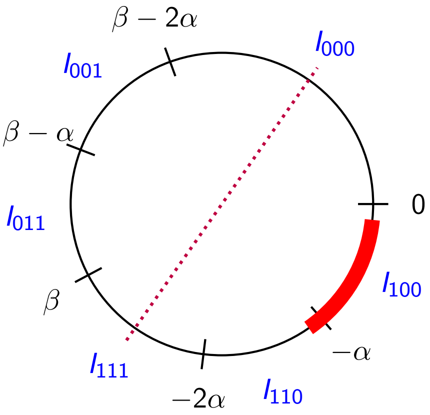

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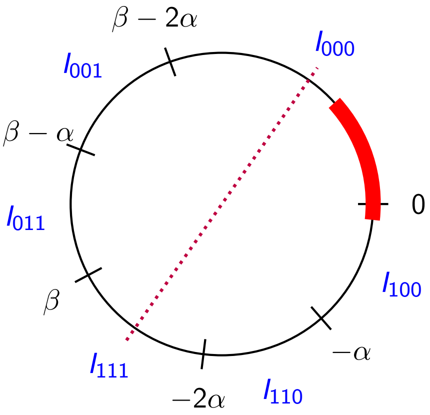

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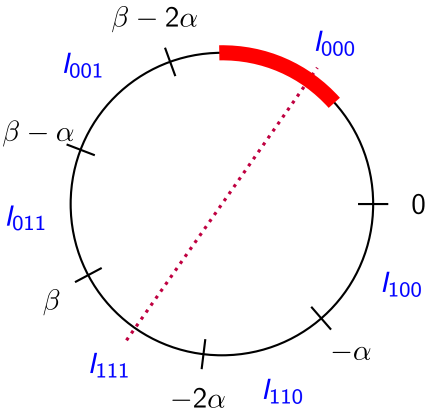

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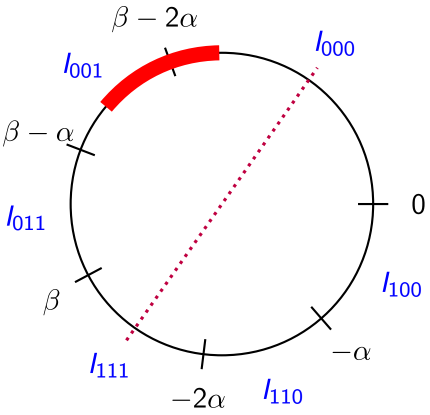

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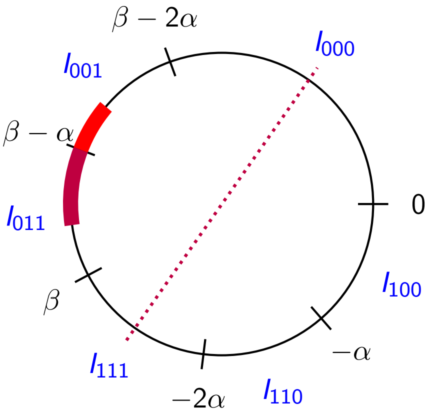

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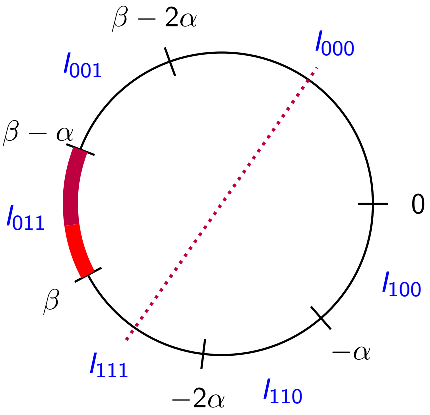

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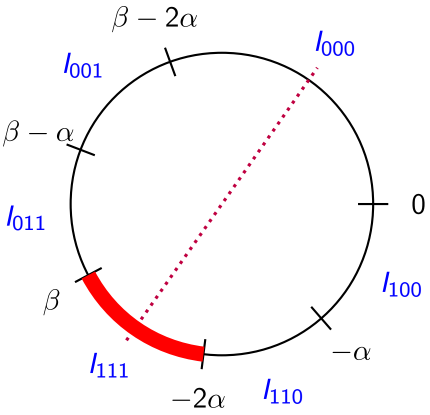

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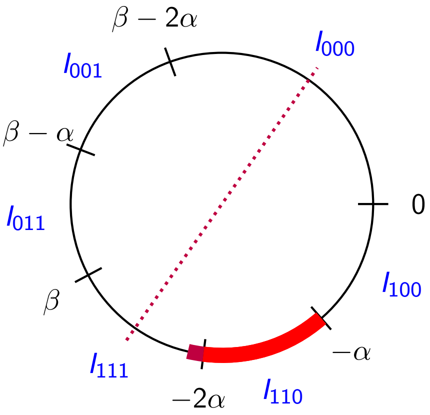

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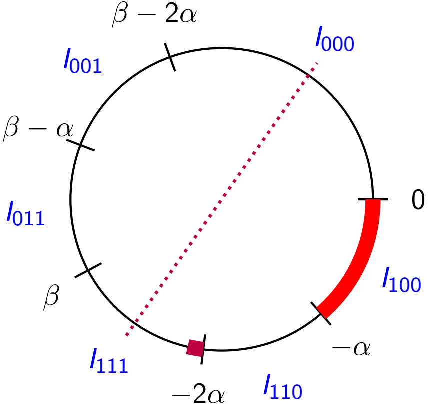

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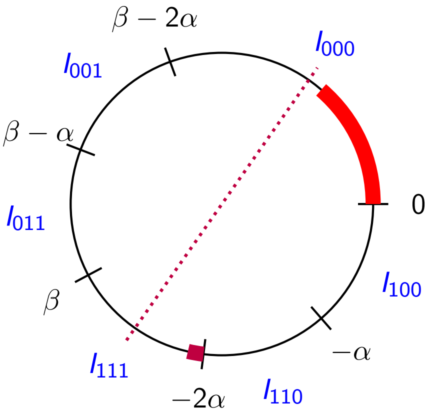

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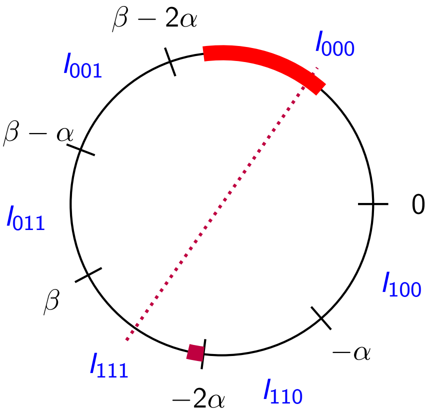

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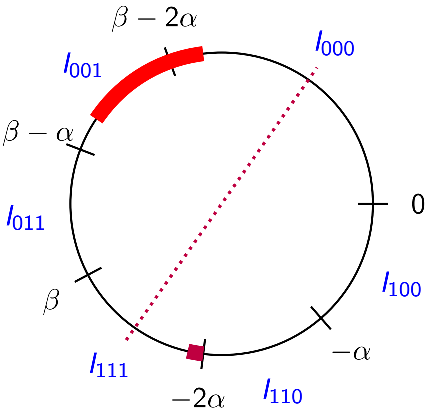

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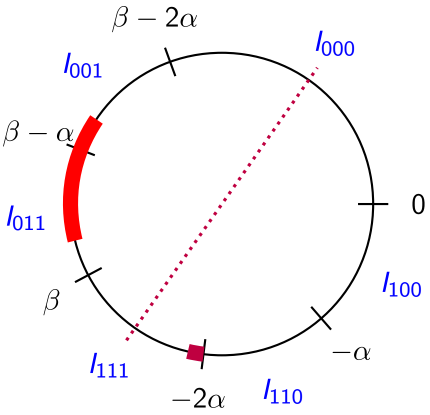

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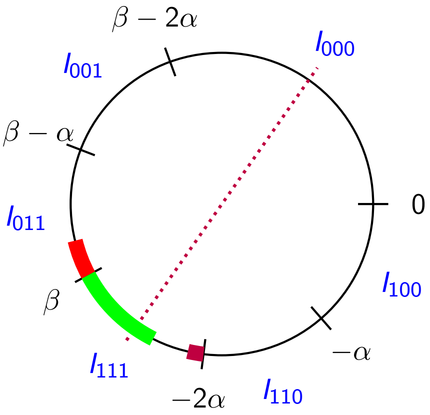

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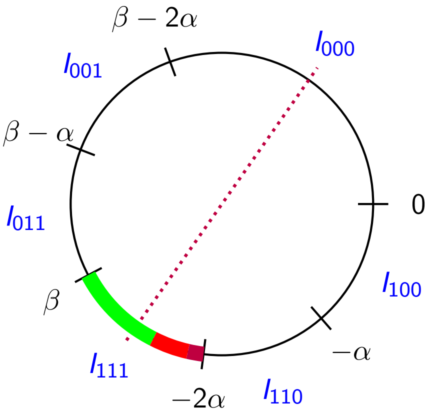

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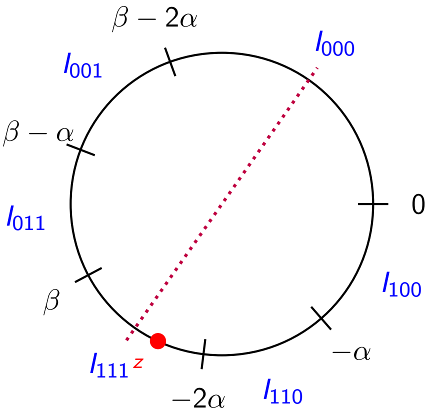

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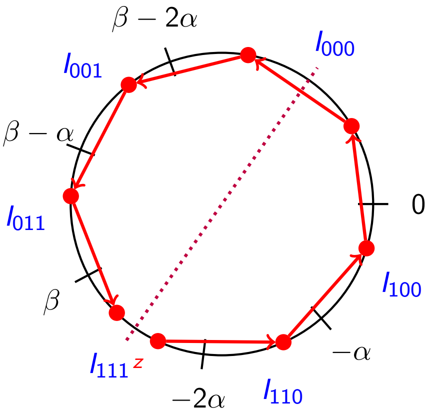

Partie 3

Conclusion

- Défaut palindromique
- Les 4 classes de complexité palindromique
- Conjecture de Hof, Knill et Simon
- Conjectures

Palindrome defect

Brlek, Hamel, Nivat, Reutenauer (2004) also defined the defect of a finite word w as

$$D(w) = |w| + 1 - |\mathrm{Pal}(w)|.$$

This definition is extended to infinite words **w** by setting $D(\mathbf{w})$ to be the maximum of the defect of its factors : it may be finite or infinite.

Palindrome complexity classes

The palindrome complexity $|\mathrm{Pal}(\mathbf{w})|$ and defect $D(\mathbf{w})$ divides the set of infinite words $\mathbf{w}$ into four classes :

$\lvert\mathrm{Pal}(\mathbf{w})\rvert$	$D(\mathbf{w})$	Examples
∞	0	Sturmian words, Coding of rotation
∞	$0 < D(\mathbf{w}) < \infty$	$(aababbaabbabaa)^\omega$
∞	∞	Thue-Morse word
finite	∞	Fixed point of $a \mapsto abb$, $b \mapsto ba$

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Conjecture (Blondin-Massé, Brlek, L., 2008)

Let $\mathbf{w}$ be the fixed point of a primitive morphism φ. If the defect of $\mathbf{w}$ is such that $0 < D(\mathbf{w}) < \infty$, then $\mathbf{w}$ is periodic.

The conjecture can also be stated for (uniformly) recurrent words in general.

Palindrome complexity classes : $|\mathrm{Pal}(\mathbf{w})| = \infty$

A substitution is in class P if there is a palindrome p and for each $\alpha \in \Sigma$ there is a palindrome q_α such that $\varphi(\alpha) = pq_\alpha$ (Hof, Knill and Simon, 1995).

Conjecture (adaptated from Hof, Knill and Simon, 1995)

Let $\mathbf{w}$ be the fixed point of a primitive substitution. Then $|\mathrm{Pal}(\mathbf{w})| = \infty$ if and only if there is a substitution φ such that $\varphi(\mathbf{w}) = \mathbf{w}$ and φ is conjugate to a class P substitution.

This conjecture was proved by B. Tan (2007) for the binary case and was generalized to f-pseudo-palindromes in the binary case for uniform morphisms (L., 2008).

The Thue-Morse word palindrome complexity class

The 12381 primitive morphisms φ on $\{a, b\}^*$ prolongable on a such that $|\varphi(ab)| \leq 11$ group as follows :

$\lvert\mathrm{Pal}(\mathbf{w})\rvert$	$D(\mathbf{w})$	First 12381 primitive morphisms
∞	0	2649 (21%)
∞	$0 < D(\mathbf{w}) < \infty$	≈ 0
∞	∞	58 (0.5%)
finite	∞	9674 (78%)

Can we characterize the Thue-Morse word palindrome complexity class ?

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Can we characterize the Thue-Morse word palindrome complexity class ?

Conjecture

Let $\mathbf{w}$ be the fixed point of a primitive substitution. Then $\mathbf{w}$ is such that $|\mathrm{Pal}(\mathbf{w})| = D(w) = \infty$ if and only if there exists a morphism φ in class P such that $\varphi(\mathbf{w}) = \mathbf{w}$ and "looks like" the Thue-Morse morphism.

The Thue-Morse word palindrome complexity class

Below are the primitive morphisms φ on $\{a, b\}^*$ prolongable on a such that $|\varphi(ab)| \leq 10$ that generates a fixed point in the Thue-Morse palindrome complexity class :

$a \mapsto ab, b \mapsto ba$	$a \mapsto aababa, b \mapsto abb$	$a \mapsto aabbabba, b \mapsto ab$
	$a \mapsto aabbaa, b \mapsto bab$	$a \mapsto abbabbaa, b \mapsto ba$
	$a \mapsto ababaa, b \mapsto bba$	$a \mapsto aabbbaa, b \mapsto bab$
$a \mapsto abbaab, b \mapsto ba$	$a \mapsto abbabb, b \mapsto bba$	$a \mapsto aababa, b \mapsto abbb$
$a \mapsto aabb, b \mapsto bbaa$	$a \mapsto abbba, b \mapsto baab$	$a \mapsto ababaa, b \mapsto bbba$
$a \mapsto abab, b \mapsto baba$	$a \mapsto abba, b \mapsto baaab$	$a \mapsto abbbba, b \mapsto baab$
$a \mapsto abba, b \mapsto baab$	$a \mapsto aab, b \mapsto baabaa$	$a \mapsto abbba, b \mapsto baaab$
$a \mapsto ab, b \mapsto baabba$	$a \mapsto aab, b \mapsto bababb$	$a \mapsto aaab, b \mapsto bababb$
	$a \mapsto aba, b \mapsto bbaabb$	$a \mapsto abba, b \mapsto baaaab$
	$a \mapsto abb, b \mapsto aababa$	$a \mapsto abbb, b \mapsto aababa$
		$a \mapsto aba, b \mapsto bbaaabb$
		$a \mapsto ab, b \mapsto aabbabba$
		$a \mapsto ab, b \mapsto baabaabb$

Useful software

This research was driven by computer exploration using the open-source mathematical software Sage [1] and its algebraic combinatorics features developed by the Sage-Combinat community [2], and in particular, F. Saliola, A. Bergeron and S. L.
The pictures have been produced using Sage and pgf/tikz.

W. A. Stein et al., *Sage Mathematics Software (Version 4.3.4)*, The Sage Development Team, 2010, http ://www.sagemath.org.

The Sage-Combinat community, Sage-Combinat : enhancing Sage as a toolbox for computer exploration in algebraic combinatorics, http ://combinat.sagemath.org, 2009.