Avancées récentes sur des questions issues des pavages du plan par translation d’une tuile

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Travail fait en collaboration avec Ariane, Alexandre et Srečko
Outline

1 Introduction
   - Discrete Figures
   - Tilings
   - Beauquier and Nivat
   - Hexagonal and Square Tiles
   - A conjecture of Brlek, Dulucq, Fédou and Provençal, 2007
   - A conjecture of Provençal and Vuillon, 2008

2 (Idea of the) Proof of the first conjecture

3 (Idea of the) Proof of the second conjecture

4 Open problems
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• Discrete plane : $\mathbb{Z}^2$

• **Definition**: A *polyomino* is a finite, 4-connected subset of the plane, without holes.
The Tiling by Translation Problem

Let $P$ be a polyomino. We say that

- $P$ tiles the plane if there exists a set $T$ of non-overlapping translated copies of $P$ that covers all the plane.
- $P$ is called a tile if it tiles the plane.

![Diagram of a tiling pattern](image-url)
The Tiling by Translation Problem

Let \( P \) be a polyomino. We say that

- \( P \) tiles the plane if there exists a set \( T \) of non-overlapping translated copies of \( P \) that covers all the plane.
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Problem

Does a given polyomino \( P \) tile the plane?
Sur les tuiles doubles carrées
\[ \Sigma = \mathbb{Z}_4 = \{0, 1, 2, 3\} \]
Freeman Chain Code

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\[ w = 0103301103301111232211233233 \]
Freeman Chain Code

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Any conjugate \( w' \) of \( w \) codes the same polyomino.

\( w \) and \( w' \) are conjugate if there exist \( u, v \in \Sigma^* \) such that \( w = uv \) and \( w' = vu \). We write \( w \equiv_{|u|} w' \).

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\(w \text{ and } w' \text{ are conjugate if there exist } u, v \in \Sigma^* \text{ such that } w = uv \text{ and } w' = vu.\) We write \(w \equiv |u| w'.\)
**Characterization** : A polyomino $P$ tiles the plane if and only if there exist $X, Y, Z \in \Sigma^*$ such that $[w] \equiv XYZ\hat{X}\hat{Y}\hat{Z}$.

$X = 0 0 1 0 3 0 1$

$\hat{X} = 3 2 1 2 3 2 2$

hexagon tiles

square tiles
There are polyominoes admitting many hexagon tilings:

A $1 \times n$ rectangle tiles the plane as an hexagon in $n - 1$ ways and as a square in only 1 way.
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A \(1 \times n\) rectangle tiles the plane as an hexagon in \(n - 1\) ways and as a square in only 1 way.
The pentomino has two distinct square factorizations:
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Conjecture (Brlek, Dulucq, Fédou, Provençal 2007)

A tile has at most 2 square factorizations.
Palindromes in Prime Double Square Tiles

Conjecture (X. Provençal and L. Vuillon, 2008 in the PK-4214)

If $XY\hat{X}\hat{Y} \equiv WZ\hat{W}\hat{Z}$ are distinct Beauquier-Nivat factorizations of a prime double square tile, then $X$, $Y$, $W$ and $Z$ are palindromes.

Note: a palindrome is a word that reads the same forward as it does backward.
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Idea of the proof: at most 2 square factorizations

**Lemma (Brlek, Fédou, Provençal, 2008)**

The factorizations $UV\hat{U}\hat{V} \equiv d_1 XY\hat{X}\hat{Y}$ of a double square tile must alternate, that is $0 < d_1 < |U| < d_1 + |X|$.

Suppose that there is a triple square tile having the following boundary:

$UV\hat{U}\hat{V} \equiv d_1 XY\hat{X}\hat{Y} \equiv d_2 WZ\hat{W}\hat{Z}$.
Examples

Suppose that $|U| = |V| = |X| = |Y| = |W| = |Z| = 3$.

If a third factorization $WZ\hat{W}\hat{Z}$ exists, then, $0 = 2$ and $1 = 3$ which is a contradiction. Hence, there is no triple square tile of perimeter 12.

Although, there are words having more than two square factorizations. An example of length 36 was provided by X. Provençal:
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Suppose that $|U| = |V| = |X| = |Y| = |W| = |Z| = 3.$

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```
 0 0 2 0
U V U V
X Y X Y
```

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\hline
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Note that the factor $221003$ is a closed path...

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Note that the factor $221003$ is a closed path...
The first differences sequence of \( w \in (\mathbb{Z}_4)^* \)

\[
\Delta w = (w_2 - w_1) \cdot (w_3 - w_2) \cdots (w_n - w_{n-1}).
\]

represents the sequence of turns of the path.

\( w = 01012223211 \)  
\( \Delta w = 1311001330 \)
First differences sequence

The first differences sequence of $w \in (\mathbb{Z}_4)^*$

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represents the sequence of turns of the path.

\[ w = 01012223211 \quad \Delta w = 1311001330 \]

We also consider $\Delta[w]$ well defined on the conjugacy classes :

$$
\Delta[w] = (w_2 - w_1) \cdot (w_3 - w_2) \cdots (w_n - w_{n-1}) \cdot (w_1 - w_n) = \Delta w \cdot (w_1 - w_n).
$$
The turning number of a path $w$ is $\mathcal{T}(w) = \frac{|\Delta w_1| - |\Delta w_3|}{4}$ and corresponds to its total curvature divided by $2\pi$ (Wikipedia). We have that

- $\mathcal{T}(w) = -\mathcal{T}(\hat{w})$ for all path $w \in \Sigma^*$
- $\mathcal{T}([w]) = \pm 1$ for all simple and closed path $w$. 
The turning number of a path \( w \) is \( \mathcal{T}(w) = \frac{|\Delta w|_1 - |\Delta w|_3}{4} \) and corresponds to its total curvature divided by \( 2\pi \) (Wikipedia). We have that

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- \( \mathcal{T}([w]) = \pm 1 \) for all simple and closed path \( w \).

For example,

\[ w = 01012223211 \]
\[ \Delta w = 1311001330 \]
\[ \mathcal{T}(w) = 1/4 \]

\[ \hat{w} = 33010003232 \]
\[ \Delta \hat{w} = 0113003313 \]
\[ \mathcal{T}(\hat{w}) = -1/4 \]
Turning number

Then, for a square tile, the sum of the four angles between $X$, $Y$, $\hat{X}$ and $\hat{Y}$ must be $2\pi$.

Lemma (Blondin-Massé, Brlek, Garon, L. 2010)

Si $XY\hat{X}\hat{Y}$ est la frontière orientée positivement d'une tuile carrée, alors

$$\Delta[XY\hat{X}\hat{Y}] = \Delta X \cdot 1 \cdot \Delta Y \cdot 1 \cdot \Delta \hat{X} \cdot 1 \cdot \Delta \hat{Y} \cdot 1.$$
Idée de la preuve : au plus 2 factorisations carrées

On s'intéresse aux moitiés du contour

\[ x = x_0 x_1 x_2 \cdots x_{n-1} = 1 \cdot \Delta U \cdot 1 \cdot \Delta V, \]
\[ y = y_0 y_1 y_2 \cdots y_{n-1} = 1 \cdot \Delta \hat{U} \cdot 1 \cdot \Delta \hat{V}. \]

One have \( x_i = y_i = 1 \) for all \( i \in I \) where

\[ I = \{ 0, d_1, d_1 + d_2, |U|, d_1 + |X|, d_1 + d_2 + |W| \} \subseteq \mathbb{Z}_n. \]
On définit trois réflexions sur $\mathbb{Z}_n$ :

\begin{align*}
s_1 : i & \mapsto |U| - i, \\
s_2 : i & \mapsto |X| + 2d_1 - i, \\
s_3 : i & \mapsto |W| + 2(d_1 + d_2) - i.
\end{align*}

**Lemma**

Soit $i \in \mathbb{Z}_n$ et $j \in \{1, 2, 3\}$ tels que $s_j$ est admissible sur $i$. Alors

- $y_i = -x_{s_j(i)}$ et $x_i = -y_{s_j(i)}$.
- Si $x_i = y_i$, alors $x_{s_j(i)} = y_{s_j(i)}$.
Idée de la preuve : au plus 2 factorisations carrées

Soit \( n = 30, \ d_1 = 3, \ d_2 = 5, \ |U| = 17, \ |X| = 17 \) et \( |W| = 15 \).
\( 1 = x_0 = -x_{s_3s_2s_1s_3s_2}(0) = -x_{17} = -1 = 3 \) une contradiction.

On a \( s_1 = s_3s_2s_1s_3s_2 \). Si \( s_3s_2s_1s_3s_2 \) est un produit admissible de réflexions sur 0, alors \( x_0 = -x_{17} \) ce qui est une contradiction. Autrement, des contradictions similaires sont obtenues.
Theorem (Blondin Massé, Brlek, Garon, L. 2010)

* A tile has *at most* 2 square factorizations.
Outline

1 Introduction
   • Discrete Figures
   • Tilings
   • Beauquier and Nivat
   • Hexagonal and Square Tiles
   • A conjecture of Brlek, Dulucq, Fédou and Provenčal, 2007
   • A conjecture of Provenčal and Vuillon, 2008

2 (Idea of the) Proof of the first conjecture

3 (Idea of the) Proof of the second conjecture

4 Open problems
Composition of tiles

The factorization $AB\hat{A}\hat{B}$ of a square $S$ allows to define the substitution

$$\varphi_S : 0 \mapsto A, 1 \mapsto B, 2 \mapsto \hat{A}, 3 \mapsto \hat{B}.$$ 

For any polyomino $P$ having boundary $w$ we define the composition

$$S \circ P := \varphi_S(w).$$

Note: This is not commutative.
Composition of tiles

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Definition

A polyomino $Q$ is prime if $Q = S \circ P$ implies that $S$ or $P$ is the unit square.
Theorem (Blondin Massé, Brlek, Garon, L. (GASCom 2010))

Every double square reduces to a composed cross pentomino.
Reduction of double square tiles

Moreover,

- The transformations $T_i$ are invertible.
- The transformations $T_i^{-1}$ preserve palindromes.

Proposition (Blondin Massé, Brlek, Garon, L. (GASCom 2010))

Let $AB\hat{A}\hat{B} \equiv XY\hat{X}\hat{Y}$ be the boundary of a double square $D$. If $D$ reduces to the prime cross pentomino, then $A$, $B$, $X$ and $Y$ are palindromes.
Questions:

- Do every prime double square reduces to the prime cross pentomino?
- Does the reduction $T_i$ preserve prime tiles?
- Does the inverse $T_i^{-1}$ preserve composed tiles?
- Do the following diagram commutes?
Let $AB\hat{A}\hat{B} \equiv XY\hat{X}\hat{Y}$ be the factorizations of a double square tile.

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In general $|w_i - 1| + |w_i + 1|$ is a period of $w_i - 1 w_i w_i + 1$.
Let $AB\hat{A}\hat{B} \equiv XY\hat{X}\hat{Y}$ be the factorizations of a double square tile.

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Let $AB\hat{A}\hat{B} \equiv XY\hat{X}\hat{Y}$ be the factorizations of a double square tile.

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$X$ $Y$ $\hat{X}$ $\hat{Y}$

$\hat{w}_6$ $\hat{w}_5$

$A$

In general

- $|w_{i-1}| + |w_{i+1}|$ is a period of $w_{i-1}w_iw_{i+1}$.

Hence we write

- $w_i = (u_iv_i)^{n_i}u_i$ where $|u_iv_i| = |w_{i-1}| + |w_{i+1}|$. 
Transformations

Let $S = (w_0, w_1, w_2, w_3, w_4, w_5, w_6, w_7)$. We define

\[ \text{SHRINK}_0(S) = (w_0(v_0u_0)^{-1}, w_1, w_2, w_3, w_4(v_4u_4)^{-1}, w_5, w_6, w_7), \]

\[ \text{SWAP}_0(S) = (\hat{w}_4, (v_1u_1)^{n_1}v_1, \hat{w}_6, (v_3u_3)^{n_3}v_3, \hat{w}_0, (v_5u_5)^{n_5}v_5, \hat{w}_2, (v_7u_7)^{n_7}v_7) \]

and

\[ \text{EXTEND}_0(S) = (w_0(v_0u_0), w_1, w_2, w_3, w_4(v_4u_4), w_5, w_6, w_7) \]

and their conjugates

- $\text{SHIFT}(S) = (w_1, w_2, w_3, w_4, w_5, w_6, w_7, w_0)$,
- $\text{SHRINK}_i(S) = \text{SHIFT}^{-i} \circ \text{SHRINK}_0 \circ \text{SHIFT}^i(S)$,
- $\text{SWAP}_i(S) = \text{SHIFT}^{-i} \circ \text{SWAP}_0 \circ \text{SHIFT}^i(S)$,
- $\text{EXTEND}_i(S) = \text{SHIFT}^{-i} \circ \text{EXTEND}_0 \circ \text{SHIFT}^i(S)$. 
Example of reduction
Idea of the proof

Let $H(w) = |w|_0 + |w|_2$ be the number of horizontal steps of the path $w$ and $V(w) = |w|_1 + |w|_3$ be its number of vertical steps.

**Lemma**

Let $S = (w_0, w_1, w_2, w_3, w_4, w_5, w_6, w_7)$ be the factorization of a double square. Then, $H(w_i) = H(w_{i+4})$ and $V(w_i) = V(w_{i+4})$. 
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**Lemma**

SWAP\(_i\), SHRINK\(_i\) and EXTEND\(_i\) commute with the composition.
Idea of the proof

Let $H(w) = |w|_0 + |w|_2$ be the number of horizontal steps of the path $w$ and $V(w) = |w|_1 + |w|_3$ be its number of vertical steps.

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**Lemma**

SWAP$_i$, SHRINK$_i$ and EXTEND$_i$ commute with the composition.

**Proposition (Blondin-Massé, L., 2010)**

Every prime double square reduces to the prime cross pentomino.

**Corollary**

If $XY\hat{X}\hat{Y} \equiv WZ\hat{W}\hat{Z}$ are distinct Beauquier-Nivat factorizations of a prime double square tile, then $X$, $Y$, $W$ and $Z$ are palindromes.
1 Introduction
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2 (Idea of the) Proof of the first conjecture

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4 Open problems
Open problems

Some problems are left open:

- Find an algorithm that decides whether a polyomino is prime.
- If $\alpha\alpha$ appears in the boundary word of a double square tile $D$, where $\alpha \in \{0, 1, 2, 3\}$, then $D$ is not prime.
- Prove that if $S \circ P$ is a square tile, then so is $P$.
- Describe the distribution and the proportion of prime square tiles of half-perimeter $n$ as $n$ goes to infinity.
- Show that SHRINK$_i$ and SWAP$_i$ are sufficient to reduce a double square tile: no need to use the more complicated L-SHRINK$_i$ and R-SHRINK$_i$ defined for limit cases.
- Extend the results to 8-connected polyominoes.
- Extend the results to continuous paths and tiles.
This research was driven by computer exploration using the open-source mathematical software Sage.

Les images de ce document ont été produites à l’aide de pgf/tikz.