On the maximal number of ways to tile the plane by translation as a square

Sébastien Labbé

Laboratoire d'Informatique, de Robotique et de Microélectronique de Montpellier
Université Montpellier 2
Laboratoire de Combinatoire et d'Informatique Mathématique
Université du Québec à Montréal

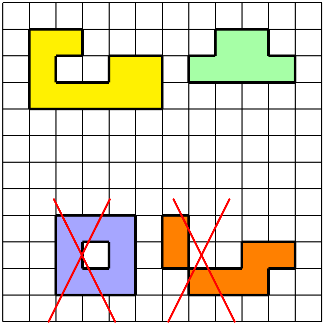
Seminar On The Interaction Between Discrete Geometry And Combinatorics On Words
Math-Info 2010
Marseille, June 8th 2010

Common work with : Alexandre Blondin Massé, Srečko Brlek et Ariane Garon

Outline

Discrete Figures and Polyominoes

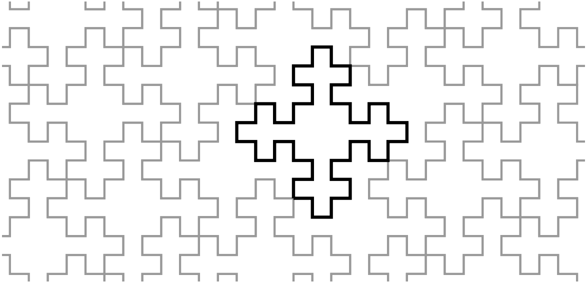
- Discrete plane : $\mathbb{Z}^2$
- **Definition** : A polyomino is a finite, 4-connected subset of the plane, without holes.



The Tiling by Translation Problem

Let P be a polyomino. We say that

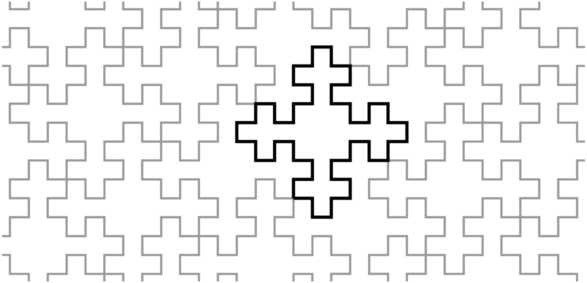
- P tiles the plane if there exists a set T of non-overlapping translated copies of P that covers all the plane.
- P is called a tile if it tiles the plane.



The Tiling by Translation Problem

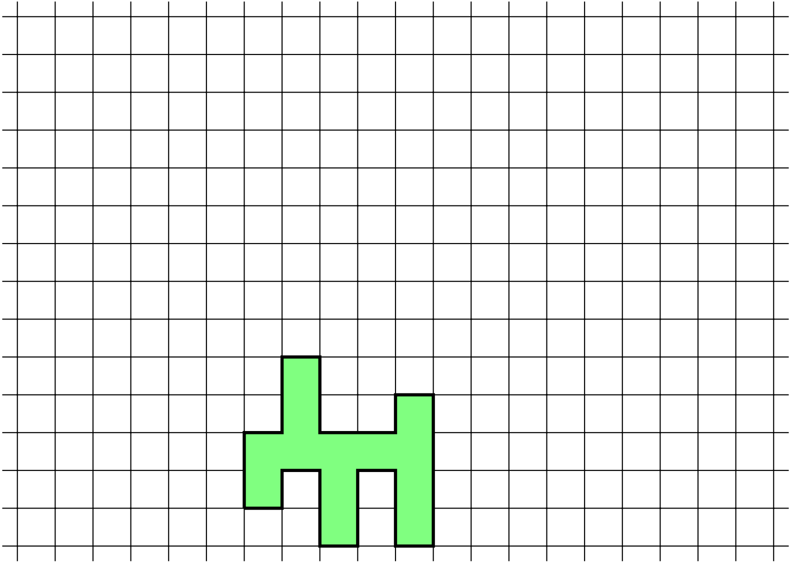
Let P be a polyomino. We say that

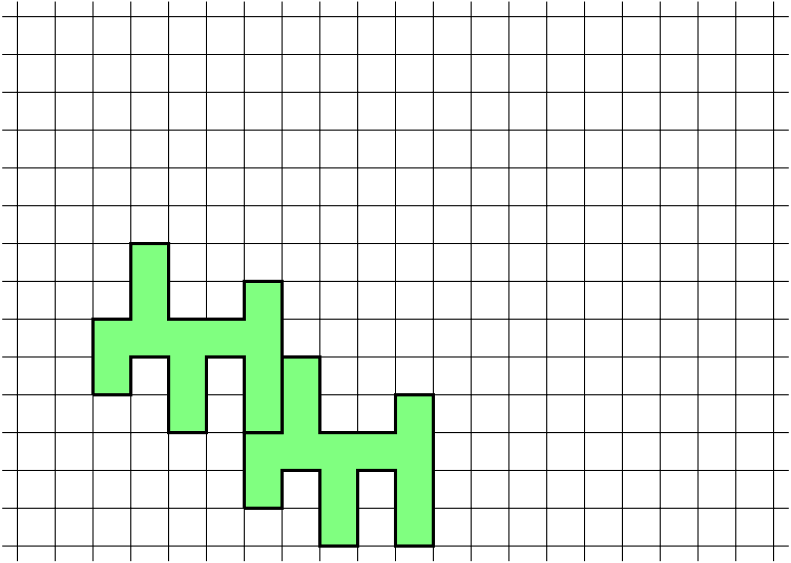
- P tiles the plane if there exists a set T of non-overlapping translated copies of P that covers all the plane.
- P is called a tile if it tiles the plane.

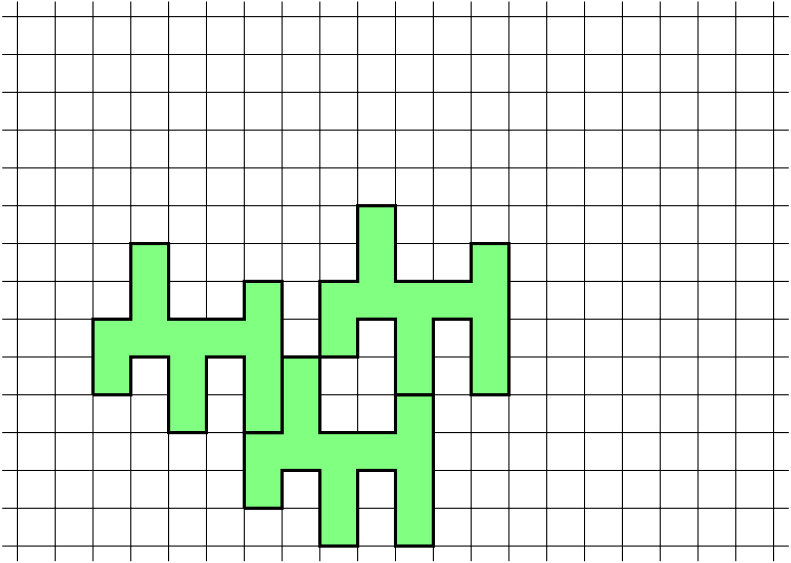


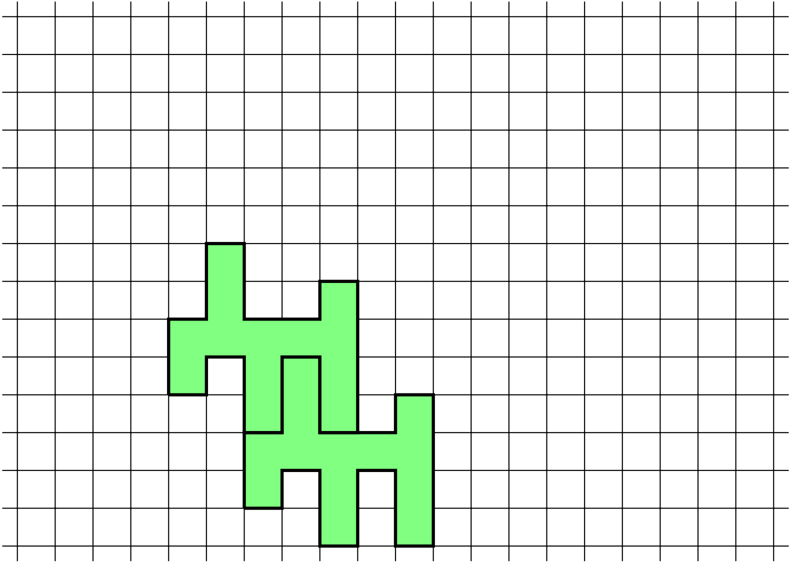
Problem

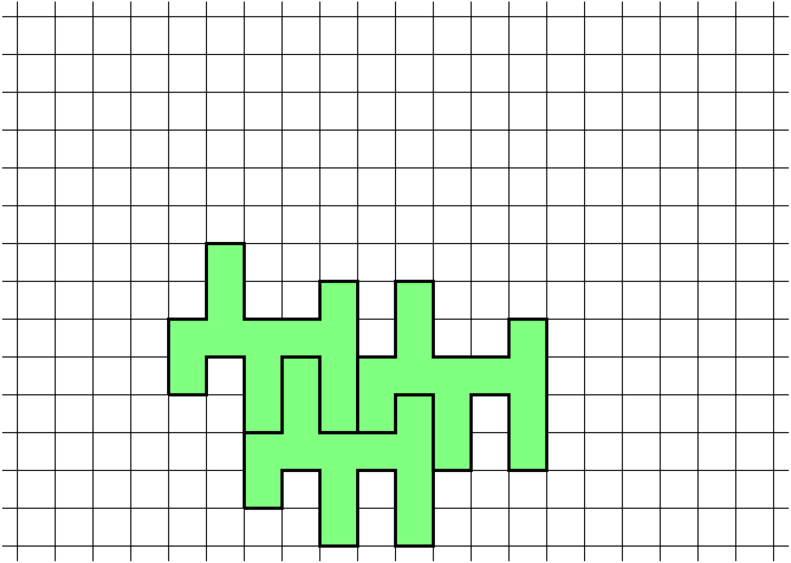
Does a given polyomino P tile the plane ?

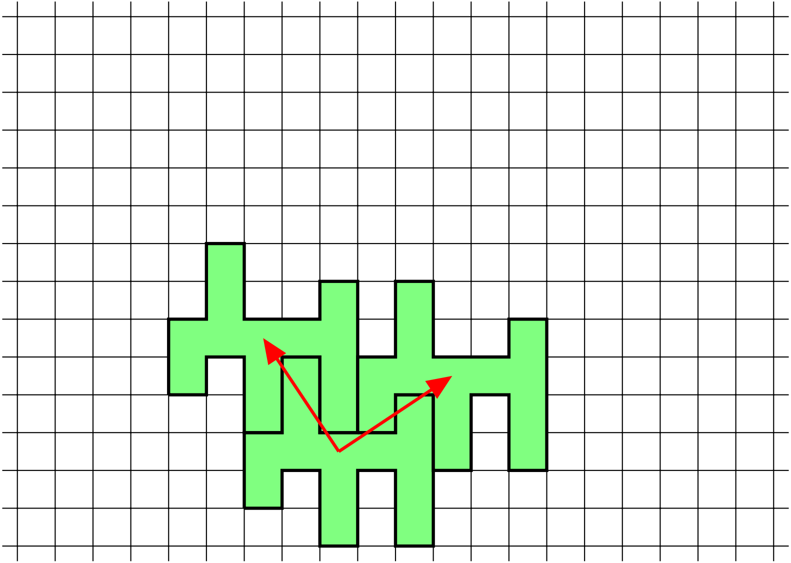


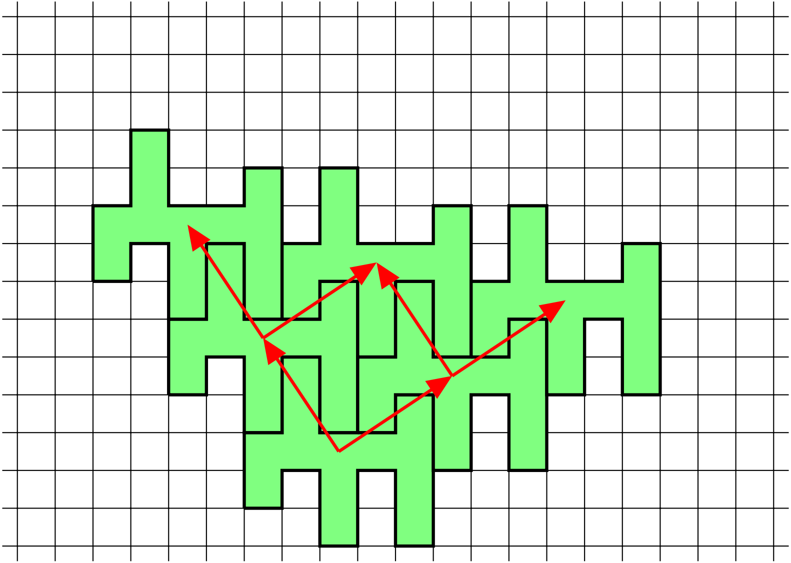


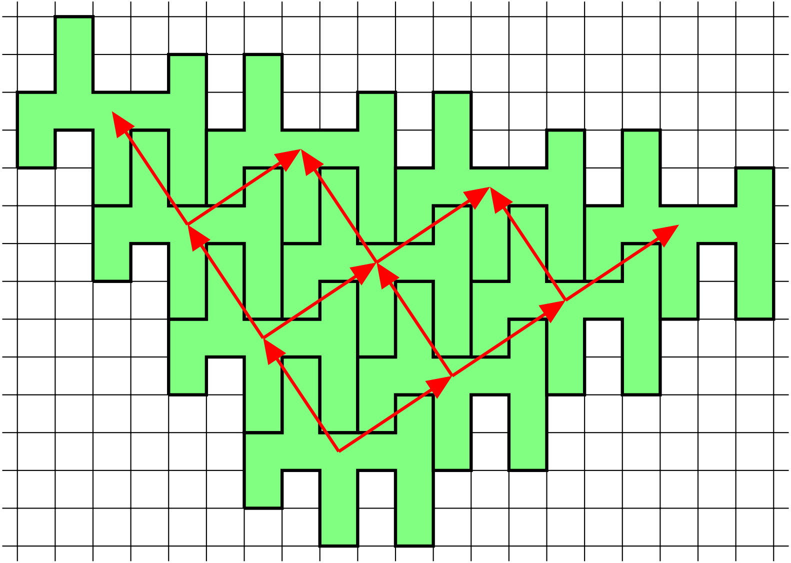


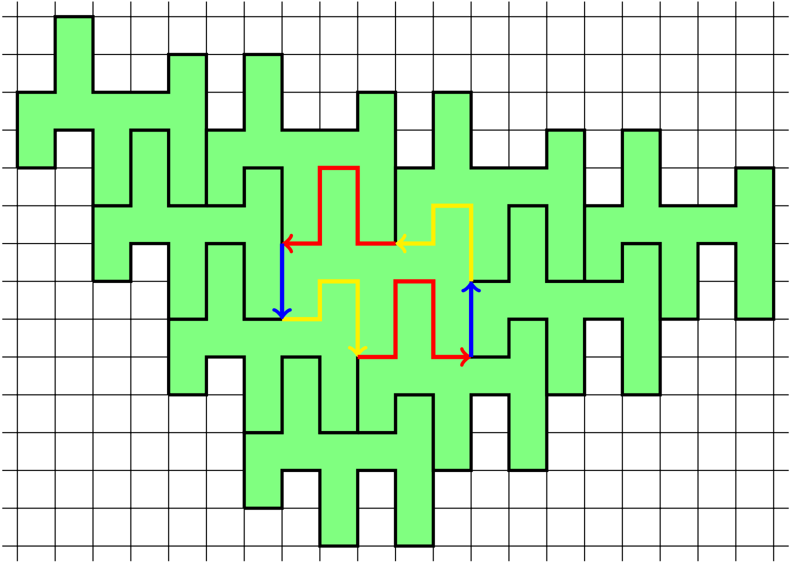






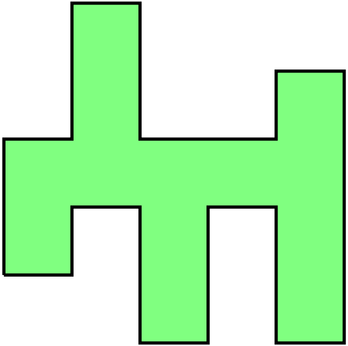


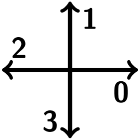




Freeman Chain Code

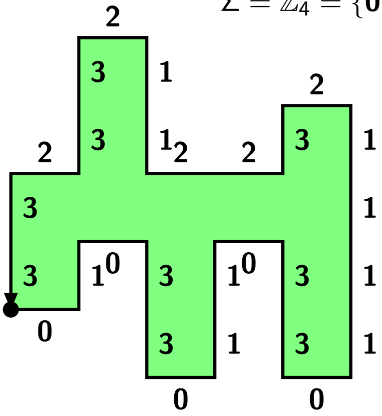
$$\Sigma = \mathbb{Z}_4 = \{\mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3}\}$$

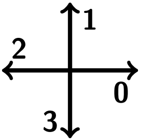




Freeman Chain Code

$$\Sigma = \mathbb{Z}_4 = \{\mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3}\}$$

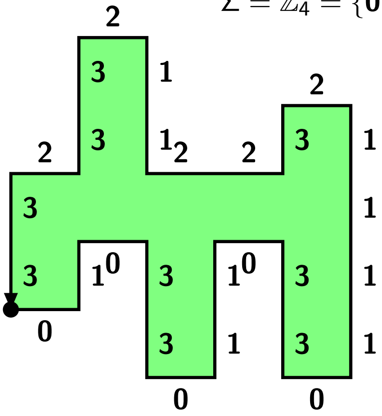


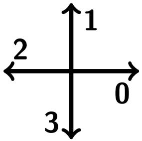


$w = \mathbf{0103301103301111232211233233}$

Freeman Chain Code

$\Sigma = \mathbb{Z}_4 = \{\mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3}\}$





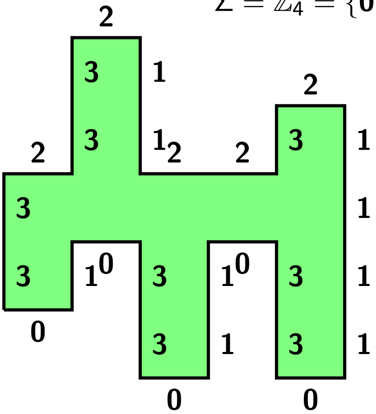
Any conjugate w' of w codes the same polyomino.

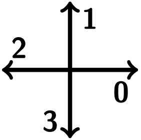
w and w' are conjugate if there exist $u, v \in \Sigma^*$ such that $w = uv$ and $w' = vu$. We write $w \equiv_{|u|} w'$.

$w = \mathbf{0103301103301111232211233233}$

Freeman Chain Code

$$\Sigma = \mathbb{Z}_4 = \{\mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3}\}$$





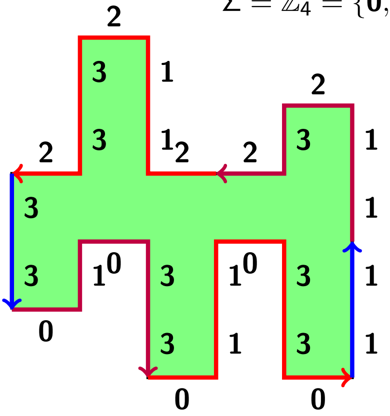
Any conjugate w' of w codes the same polyomino.

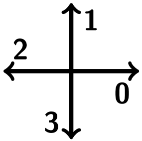
w and w' are conjugate if there exist $u, v \in \Sigma^*$ such that $w = uv$ and $w' = vu$. We write $w \equiv_{|u|} w'$.

$$[w] \equiv \mathbf{0103301103301111232211233233}$$

Freeman Chain Code

$\Sigma = \mathbb{Z}_4 = \{\mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3}\}$



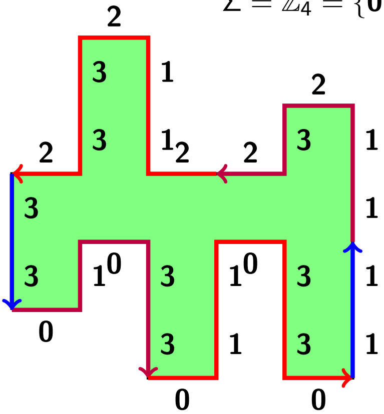


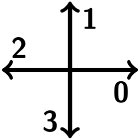
Any conjugate w' of w codes the same polyomino.

w and w' are conjugate if there exist $u, v \in \Sigma^*$ such that $w = uv$ and $w' = vu$. We write $w \equiv_{|u|} w'$.

$[w] \equiv \mathbf{0103301103301111232211233233}$

$\Sigma = \mathbb{Z}_4 = \{\mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3}\}$





Any conjugate w' of w codes the same polyomino.

w and w' are conjugate if there exist $u, v \in \Sigma^*$ such that $w = uv$ and $w' = vu$. We write $w \equiv_{|u|} w'$.

$[w] \equiv$

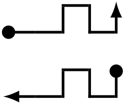
01033	0110330	11	11232	2112332	33
X	Y	Z	$\widehat{X}$	$\widehat{Y}$	$\widehat{Z}$

Beauquier and Nivat (1991)

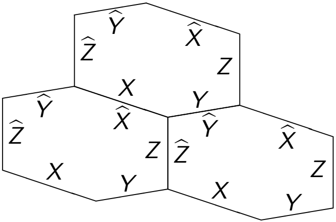
Characterization : A polyomino P tiles the plane if and only if there exist $X, Y, Z \in \Sigma^*$ such that $[w] \equiv XYZ\widehat{X}\widehat{Y}\widehat{Z}$.

$X = \mathbf{0\ 0\ 1\ 0\ 3\ 0\ 1}$

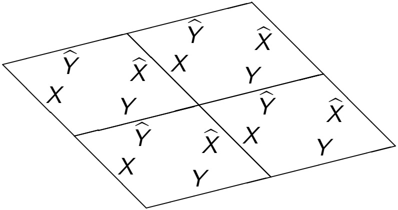
$\widehat{X} = \mathbf{3\ 2\ 1\ 2\ 3\ 2\ 2}$



hexagon tiles



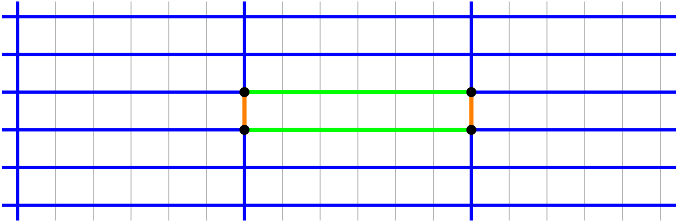
square tiles



Maurits Cornelis Escher (1898-1972). Hexagonal tiling. Square tiling.

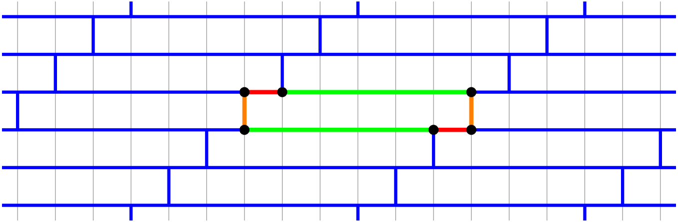
Hexagonal Tilings

There are polyominoes admitting many hexagon tilings :



A $1 \times n$ rectangle tiles the plane as an hexagon in $n-1$ ways and as a square in only 1 way.

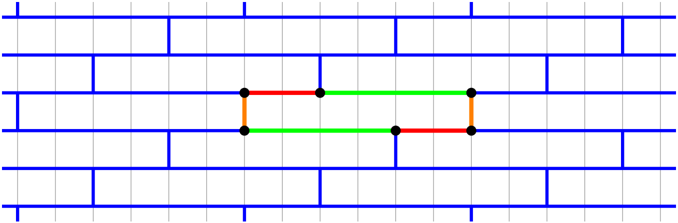
There are polyominoes admitting many hexagon tilings :



A $1 \times n$ rectangle tiles the plane as an hexagon in $n-1$ ways and as a square in only 1 way.

Hexagonal Tilings

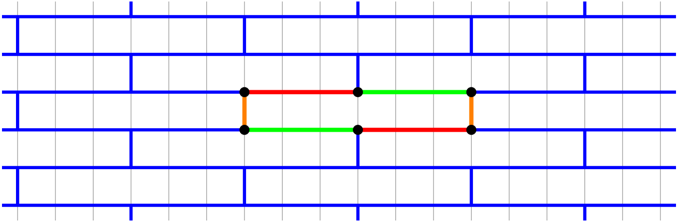
There are polyominoes admitting many hexagon tilings :



A $1 \times n$ rectangle tiles the plane as an hexagon in $n-1$ ways and as a square in only 1 way.

Hexagonal Tilings

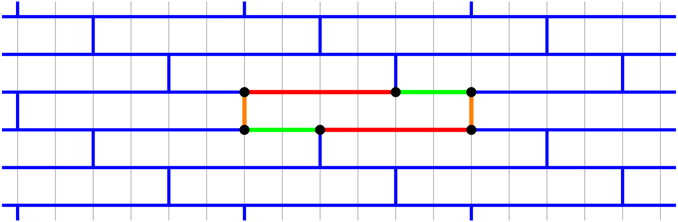
There are polyominoes admitting many hexagon tilings :



A $1 \times n$ rectangle tiles the plane as an hexagon in $n-1$ ways and as a square in only 1 way.

Hexagonal Tilings

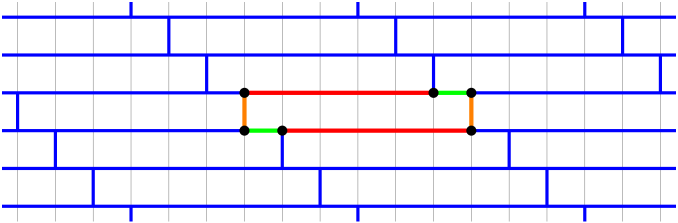
There are polyominoes admitting many hexagon tilings :



A $1 \times n$ rectangle tiles the plane as an hexagon in $n - 1$ ways and as a square in only 1 way.

Hexagonal Tilings

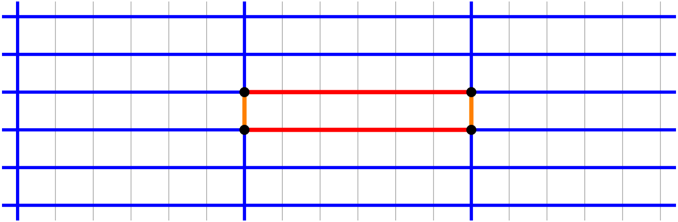
There are polyominoes admitting many hexagon tilings :



A $1 \times n$ rectangle tiles the plane as an hexagon in $n-1$ ways and as a square in only 1 way.

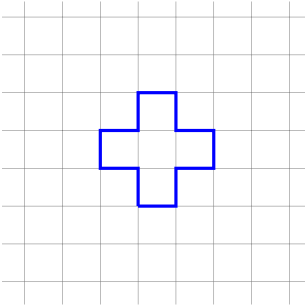
Hexagonal Tilings

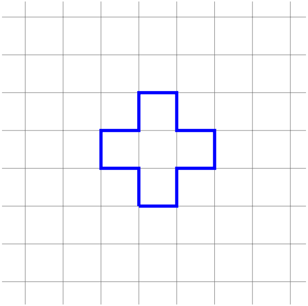
There are polyominoes admitting many hexagon tilings :



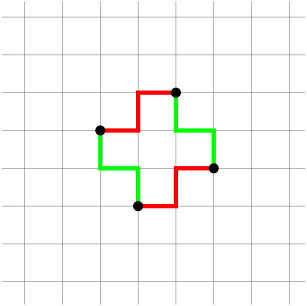
A $1 \times n$ rectangle tiles the plane as an hexagon in $n-1$ ways and as a square in only 1 way.

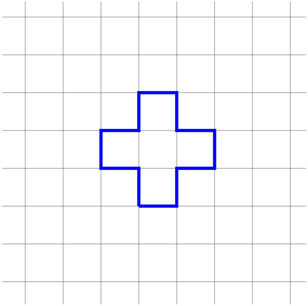
The pentamino has two distinct square factorizations :





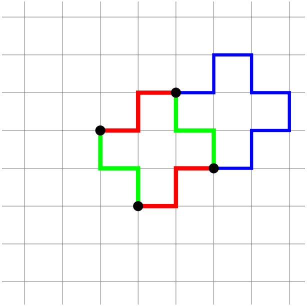
The pentamino has two distinct square factorizations :

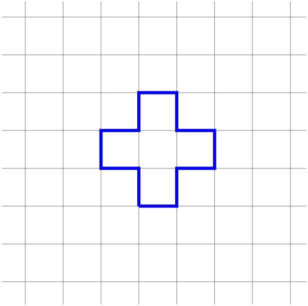




Square Tilings

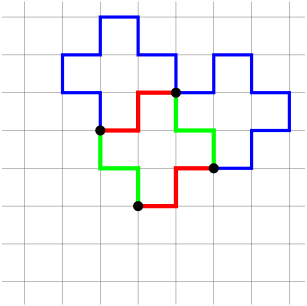
The pentamino has two distinct square factorizations :

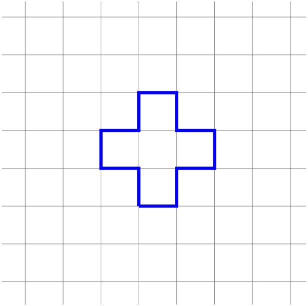




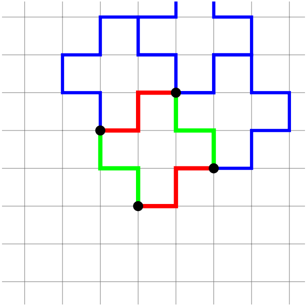
Square Tilings

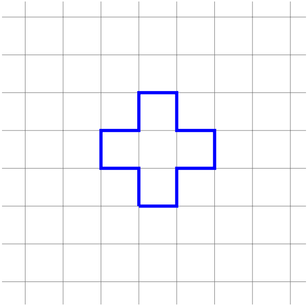
The pentamino has two distinct square factorizations :





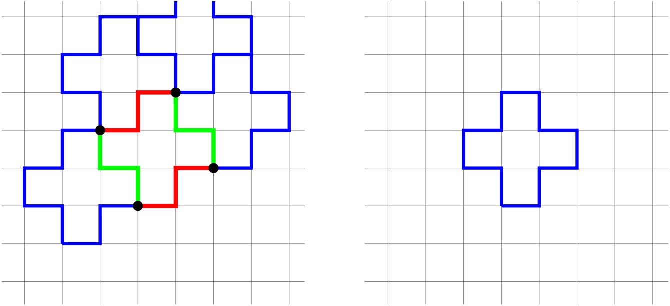
The pentamino has two distinct square factorizations :





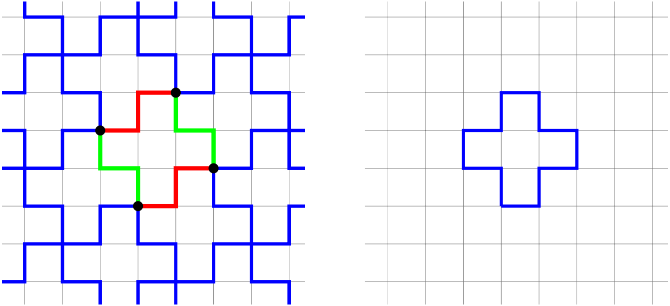
Square Tilings

The pentamino has two distinct square factorizations :



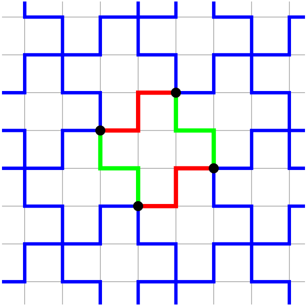
Square Tilings

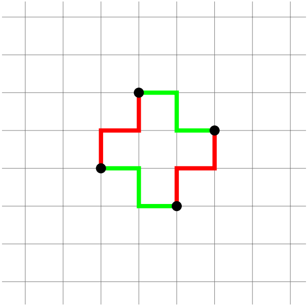
The pentamino has two distinct square factorizations :



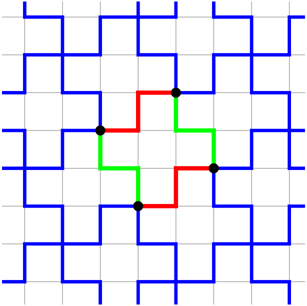
Square Tilings

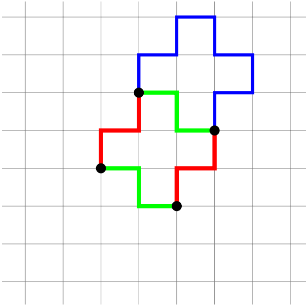
The pentamino has two distinct square factorizations :





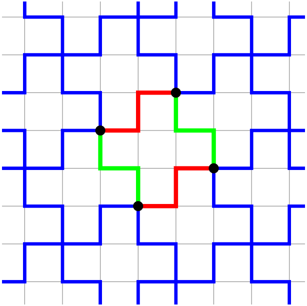
The pentamino has two distinct square factorizations :

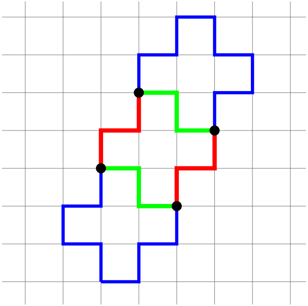




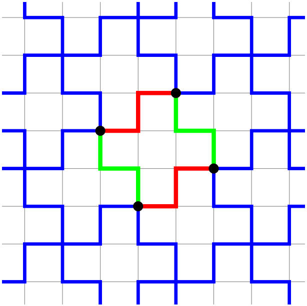
Square Tilings

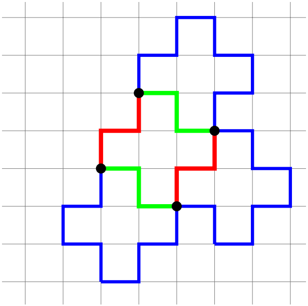
The pentamino has two distinct square factorizations :





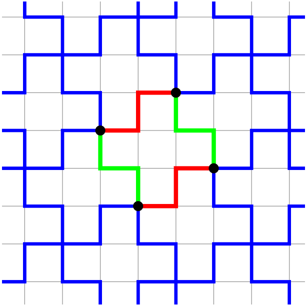
The pentamino has two distinct square factorizations :

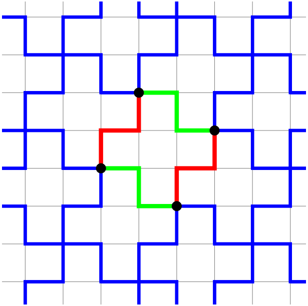




Square Tilings

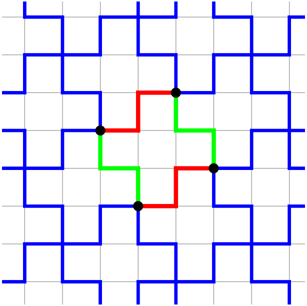
The pentamino has two distinct square factorizations :

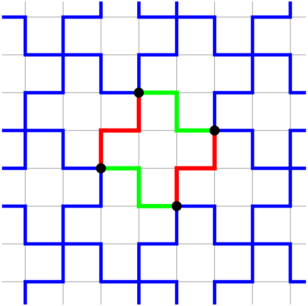




Square Tilings

The pentamino has two distinct square factorizations :





Conjecture (Brlek, Dulucq, Fédou, Provençal 2007)

A tile has at most 2 square factorizations.

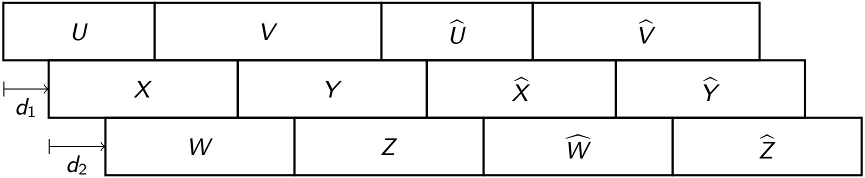
Preuve

Supposons qu'il existe une tuile triple carrée dont la frontière s'écrit :

$$UV\widehat{U}\widehat{V} \equiv_{d_1} XY\widehat{X}\widehat{Y} \equiv_{d_2} WZ\widehat{W}\widehat{Z}.$$

Lemma (Brlek, Fédou, Provençal, 2008)

Les deux factorisations $UV\widehat{U}\widehat{V} \equiv_{d_1} XY\widehat{X}\widehat{Y}$ d'une tuile double carrée doivent alterner c'est-à-dire que $0 < d_1 < |U| < d_1 + |X|$.



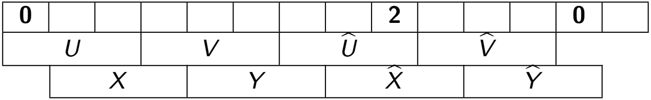
Examples

Suppose that $|U| = |V| = |X| = |Y| = |W| = |Z| = 3$.

<table>
<tr><td>0</td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td>0</td><td></td></tr>
<tr><td colspan="3">U</td><td colspan="3">V</td><td colspan="3">$\widehat{U}$</td><td colspan="3">$\widehat{V}$</td><td colspan="2"></td></tr>
<tr><td></td><td colspan="3">X</td><td colspan="3">Y</td><td colspan="3">$\widehat{X}$</td><td colspan="3">$\widehat{Y}$</td><td></td></tr>
</table>

Examples

Suppose that $|U| = |V| = |X| = |Y| = |W| = |Z| = 3$.



Examples

Suppose that $|U| = |V| = |X| = |Y| = |W| = |Z| = 3$.

<table>
<tr><td>0</td><td></td><td>0</td><td></td><td></td><td></td><td></td><td></td><td>2</td><td></td><td></td><td></td><td>0</td><td></td></tr>
<tr><td colspan="3">U</td><td colspan="3">V</td><td colspan="3">$\widehat{U}$</td><td colspan="3">$\widehat{V}$</td><td colspan="2"></td></tr>
<tr><td></td><td colspan="3">X</td><td colspan="3">Y</td><td colspan="3">$\widehat{X}$</td><td colspan="3">$\widehat{Y}$</td><td></td></tr>
</table>

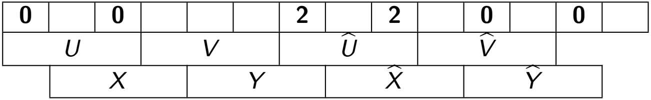
Examples

Suppose that $|U| = |V| = |X| = |Y| = |W| = |Z| = 3$.

<table>
<tr><td>0</td><td></td><td>0</td><td></td><td></td><td></td><td>2</td><td></td><td>2</td><td></td><td></td><td></td><td>0</td><td></td></tr>
<tr><td colspan="3">U</td><td colspan="3">V</td><td colspan="3">$\widehat{U}$</td><td colspan="3">$\widehat{V}$</td><td colspan="2"></td></tr>
<tr><td></td><td colspan="3">X</td><td colspan="3">Y</td><td colspan="3">$\widehat{X}$</td><td colspan="3">$\widehat{Y}$</td><td></td></tr>
</table>

Examples

Suppose that $|U| = |V| = |X| = |Y| = |W| = |Z| = 3.$



Examples

Suppose that $|U| = |V| = |X| = |Y| = |W| = |Z| = 3.$

<table>
<tr><td>0</td><td></td><td>0</td><td></td><td>2</td><td></td><td>2</td><td></td><td>2</td><td></td><td>0</td><td></td><td>0</td><td></td></tr>
<tr><td colspan="3">U</td><td colspan="3">V</td><td colspan="3">$\widehat{U}$</td><td colspan="3">$\widehat{V}$</td><td colspan="2"></td></tr>
<tr><td></td><td colspan="3">X</td><td colspan="3">Y</td><td colspan="3">$\widehat{X}$</td><td colspan="3">$\widehat{Y}$</td><td></td></tr>
</table>

Examples

Suppose that $|U| = |V| = |X| = |Y| = |W| = |Z| = 3$.

<table>
<tr><td>0</td><td>1</td><td>0</td><td></td><td>2</td><td></td><td>2</td><td></td><td>2</td><td></td><td>0</td><td></td><td>0</td><td>1</td></tr>
<tr><td colspan="3">U</td><td colspan="3">V</td><td colspan="3">$\widehat{U}$</td><td colspan="3">$\widehat{V}$</td><td colspan="2"></td></tr>
<tr><td></td><td colspan="3">X</td><td colspan="3">Y</td><td colspan="3">$\widehat{X}$</td><td colspan="3">$\widehat{Y}$</td><td></td></tr>
</table>

Examples

Suppose that $|U| = |V| = |X| = |Y| = |W| = |Z| = 3$.

<table>
<tr><td>0</td><td>1</td><td>0</td><td></td><td>2</td><td></td><td>2</td><td>3</td><td>2</td><td></td><td>0</td><td></td><td>0</td><td>1</td></tr>
<tr><td colspan="3">U</td><td colspan="3">V</td><td colspan="3">$\widehat{U}$</td><td colspan="3">$\widehat{V}$</td><td colspan="2"></td></tr>
<tr><td></td><td colspan="3">X</td><td colspan="3">Y</td><td colspan="3">$\widehat{X}$</td><td colspan="3">$\widehat{Y}$</td><td></td></tr>
</table>

Examples

Suppose that $|U| = |V| = |X| = |Y| = |W| = |Z| = 3$.

<table>
<tr><td>0</td><td>1</td><td>0</td><td>1</td><td>2</td><td></td><td>2</td><td>3</td><td>2</td><td></td><td>0</td><td></td><td>0</td><td>1</td></tr>
<tr><td colspan="3">U</td><td colspan="3">V</td><td colspan="3">$\widehat{U}$</td><td colspan="3">$\widehat{V}$</td><td colspan="2"></td></tr>
<tr><td></td><td colspan="3">X</td><td colspan="3">Y</td><td colspan="3">$\widehat{X}$</td><td colspan="3">$\widehat{Y}$</td><td></td></tr>
</table>

Examples

Suppose that $|U| = |V| = |X| = |Y| = |W| = |Z| = 3$.

<table>
<tr><td>0</td><td>1</td><td>0</td><td>1</td><td>2</td><td></td><td>2</td><td>3</td><td>2</td><td></td><td>0</td><td>3</td><td>0</td><td>1</td></tr>
<tr><td colspan="3">U</td><td colspan="3">V</td><td colspan="3">$\widehat{U}$</td><td colspan="3">$\widehat{V}$</td><td></td><td></td></tr>
<tr><td></td><td colspan="3">X</td><td colspan="3">Y</td><td colspan="3">$\widehat{X}$</td><td colspan="3">$\widehat{Y}$</td><td></td></tr>
</table>

Examples

Suppose that $|U| = |V| = |X| = |Y| = |W| = |Z| = 3$.

0	**1**	**0**	**1**	**2**	**1**	**2**	**3**	**2**		**0**	**3**	**0**	**1**
	U			V			$\widehat{U}$			$\widehat{V}$			
		X			Y			$\widehat{X}$			$\widehat{Y}$		

Examples

Suppose that $|U| = |V| = |X| = |Y| = |W| = |Z| = 3$.

<table>
<tr><td>0</td><td>1</td><td>0</td><td>1</td><td>2</td><td>1</td><td>2</td><td>3</td><td>2</td><td>3</td><td>0</td><td>3</td><td>0</td><td>1</td></tr>
<tr><td colspan="3">U</td><td colspan="3">V</td><td colspan="3">$\widehat{U}$</td><td colspan="3">$\widehat{V}$</td><td colspan="2"></td></tr>
<tr><td></td><td colspan="3">X</td><td colspan="3">Y</td><td colspan="3">$\widehat{X}$</td><td colspan="3">$\widehat{Y}$</td><td></td></tr>
</table>

Examples

Suppose that $|U| = |V| = |X| = |Y| = |W| = |Z| = 3$.

<table>
<tr><td>0</td><td>1</td><td>0</td><td>1</td><td>2</td><td>1</td><td>2</td><td>3</td><td>2</td><td>3</td><td>0</td><td>3</td><td>0</td><td>1</td></tr>
<tr><td colspan="3">U</td><td colspan="3">V</td><td colspan="3">$\widehat{U}$</td><td colspan="3">$\widehat{V}$</td><td colspan="2"></td></tr>
<tr><td></td><td colspan="3">X</td><td colspan="3">Y</td><td colspan="3">$\widehat{X}$</td><td colspan="3">$\widehat{Y}$</td><td></td></tr>
<tr><td colspan="2"></td><td colspan="3">W</td><td colspan="3">Z</td><td colspan="3">$\widehat{W}$</td><td colspan="3">$\widehat{Z}$</td></tr>
</table>

Examples

Suppose that $|U| = |V| = |X| = |Y| = |W| = |Z| = 3$.

0	**1**	**0**	**1**	**2**	**1**	**2**	**3**	**2**	**3**	**0**	**3**	**0**	**1**
	U			V			$\widehat{U}$			$\widehat{V}$			
		X			Y			$\widehat{X}$			$\widehat{Y}$		
			W			Z			$\widehat{W}$			$\widehat{Z}$	

If a third factorization $WZ\widehat{W}\widehat{Z}$ exists, then, $\mathbf{0} = \mathbf{2}$ and $\mathbf{1} = \mathbf{3}$ which is a contradiction. Hence, there is no triple square tile of perimeter 12.

Examples

Suppose that $|U| = |V| = |X| = |Y| = |W| = |Z| = 3$.

0	**1**	**0**	**1**	**2**	**1**	**2**	**3**	**2**	**3**	**0**	**3**	**0**	**1**
U			V			$\widehat{U}$			$\widehat{V}$				
	X			Y			$\widehat{X}$			$\widehat{Y}$			
		W			Z			$\widehat{W}$			$\widehat{Z}$		

If a third factorization $WZ\widehat{W}\widehat{Z}$ exists, then, $\mathbf{0} = \mathbf{2}$ and $\mathbf{1} = \mathbf{3}$ which is a contradiction. Hence, there is no triple square tile of perimeter 12.

Although, there are words having more than two square factorizations. An example of length 36 was provided by X. Provençal :

0	**0**	**122**	**10012**	**21001**	**221**	**0**	**0**	**322**	**30032**	**23003**	**223**	
U			V			$\widehat{U}$			$\widehat{V}$			
	X			Y			$\widehat{X}$			$\widehat{Y}$		
		W			Z			$\widehat{W}$			$\widehat{Z}$	

Examples

Suppose that $|U| = |V| = |X| = |Y| = |W| = |Z| = 3$.

0	**1**	**0**	**1**	**2**	**1**	**2**	**3**	**2**	**3**	**0**	**3**	**0**	**1**
U			V			$\widehat{U}$			$\widehat{V}$				
	X			Y			$\widehat{X}$			$\widehat{Y}$			
		W			Z			$\widehat{W}$			$\widehat{Z}$		

If a third factorization $WZ\widehat{W}\widehat{Z}$ exists, then, $\mathbf{0} = \mathbf{2}$ and $\mathbf{1} = \mathbf{3}$ which is a contradiction. Hence, there is no triple square tile of perimeter 12.

Although, there are words having more than two square factorizations. An example of length 36 was provided by X. Provençal :

0	**0**	**122**	**10012**	**21001**	**221**	**0**	**0**	**322**	**30032**	**23003**	**223**	
U			V			$\widehat{U}$			$\widehat{V}$			
	X			Y			$\widehat{X}$			$\widehat{Y}$		
		W		Z				$\widehat{W}$			$\widehat{Z}$	

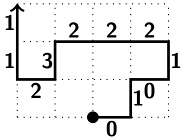
Note that the factor **221003** is a closed path...

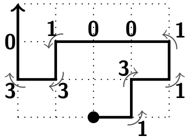
Suite des différences successives

La suite des différences successives de $w \in \Sigma^*$ est

$$\Delta w = (w_2 - w_1) \cdot (w_3 - w_2) \cdots (w_n - w_{n-1}).$$

Elle représente la suite des virages d'un chemin.





$w = \mathbf{01012223211}$ $\qquad$ $\Delta w = \mathbf{1311001330}$

On considère aussi $\Delta[w]$ bien définie sur les classes de conjugaison :

$$\Delta[w] = (w_2 - w_1) \cdot (w_3 - w_2) \cdots (w_n - w_{n-1}) \cdot (w_1 - w_n) = \Delta w \cdot (w_1 - w_n).$$

Turning number

Le turning number d'un chemin w est

$$\mathcal{T}(w) = \frac{|\Delta w|_{\mathbf{1}} - |\Delta w|_{\mathbf{3}}}{4}$$

et correspond à sa courbure totale divisée par 2π. On a

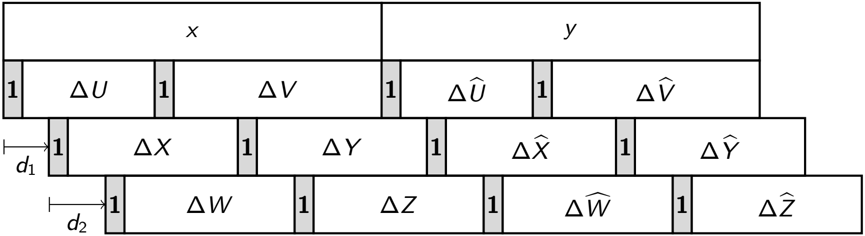
- $\mathcal{T}(w) = -\mathcal{T}(\widehat{w})$ pour tout chemin $w \in \Sigma^*$
- $\mathcal{T}([w]) = \pm 1$ pour tout chemin simple et fermé w.

Lemma

Si $XY\widehat{X}\widehat{Y}$ est la frontière orientée positivement d'une tuile carrée, alors

$$\Delta[XY\widehat{X}\widehat{Y}] = \Delta X \cdot \mathbf{1} \cdot \Delta Y \cdot \mathbf{1} \cdot \Delta\widehat{X} \cdot \mathbf{1} \cdot \Delta\widehat{Y} \cdot \mathbf{1}.$$

Preuve



On s'intéresse aux moitiés du contour

$$\begin{aligned} x &= x_0 x_1 x_2 \cdots x_{n-1} &= \mathbf{1} \cdot \Delta U \cdot \mathbf{1} \cdot \Delta V, \\ y &= y_0 y_1 y_2 \cdots y_{n-1} &= \mathbf{1} \cdot \Delta \widehat{U} \cdot \mathbf{1} \cdot \Delta \widehat{V}. \end{aligned}$$

One have $x_i = y_i = \mathbf{1}$ for all $i \in I$ where
$I = \{0, d_1, d_1 + d_2, |U|, d_1 + |X|, d_1 + d_2 + |W|\} \subseteq \mathbb{Z}_n$.

Preuve

On définit trois réflexions sur $\mathbb{Z}_n$:

$$\begin{aligned} s_1 : i &\mapsto |U| - i, \\ s_2 : i &\mapsto |X| + 2d_1 - i, \\ s_3 : i &\mapsto |W| + 2(d_1 + d_2) - i. \end{aligned}$$

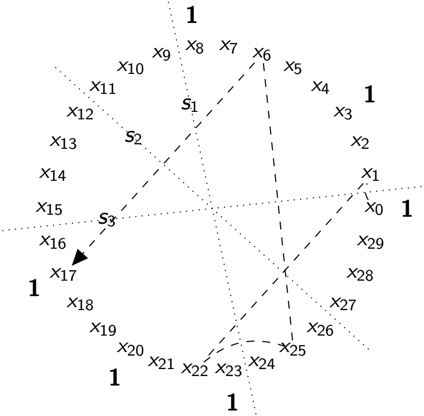
Lemma

Soit $i \in \mathbb{Z}_n$ et $j \in \{1, 2, 3\}$ tels que s_j est admissible sur i. Alors

- $y_i = -x_{s_j(i)}$ *et* $x_i = -y_{s_j(i)}$.
- *Si* $x_i = y_i$, *alors* $x_{s_j(i)} = y_{s_j(i)}$.

Preuve

Soit $n = 30$, $d_1 = 3$, $d_2 = 5$, $|U| = 17$, $|X| = 17$ et $|W| = 15$.
$\mathbf{1} = x_0 = -x_{s_3 s_2 s_1 s_3 s_2(0)} = -x_{17} = -\mathbf{1} = \mathbf{3}$ une contradiction.



On a $s_1 = s_3 s_2 s_1 s_3 s_2$. Si $s_3 s_2 s_1 s_3 s_2$ est un produit admissible de réflexions sur 0, alors $x_0 = -x_{17}$ ce qui est une contradiction. Autrement, des contradictions similaires sont obtenues.

Théorème

Theorem (Blondin-Massé, Brlek, Garon, L.)

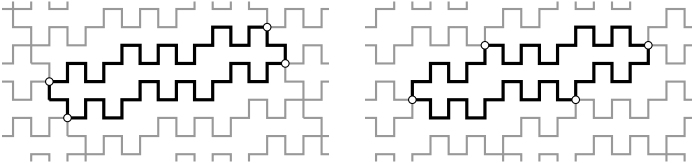
Un polyomino pave le plan à la manière d'un carré en au plus 2 façons.

Prime Double Square Tiles

Conjecture (X. Provençal and L. Vuillon, 2008)

If $XY\widehat{X}\widehat{Y}$ describes the contour of a prime double square tile, then both X and Y are palindromes.

Note : a palindrome is a word that reads the same forward as it does backward.



Crédits

- This research was driven by computer exploration using the open-source mathematical software Sage and its algebraic combinatorics features developed by the Sage-Combinat community, and in particular, F. Saliola, A. Bergeron and S. Labbé.
- Les images de ce document ont été produites à l'aide de pgf/tikz.

Liens :

- sagemath.org
- combinat.sagemath.org