Christoffel and Fibonacci Tiles

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   - Tilings
   - Beauquier and Nivat
   - Hexagonal and Square Tilings
2. Double Square Tiles
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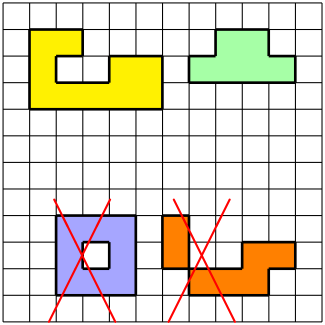
Discrete Figures and Polyominoes

- Discrete plane : $\mathbb{Z}^2$
- **Definition** : A polyomino is a finite, 4-connected subset of the plane, without holes.



The Tiling by Translation Problem

Let P be a polyomino. We say that

- P tiles the plane if there exists a set T of non-overlapping translated copies of P that covers all the plane.
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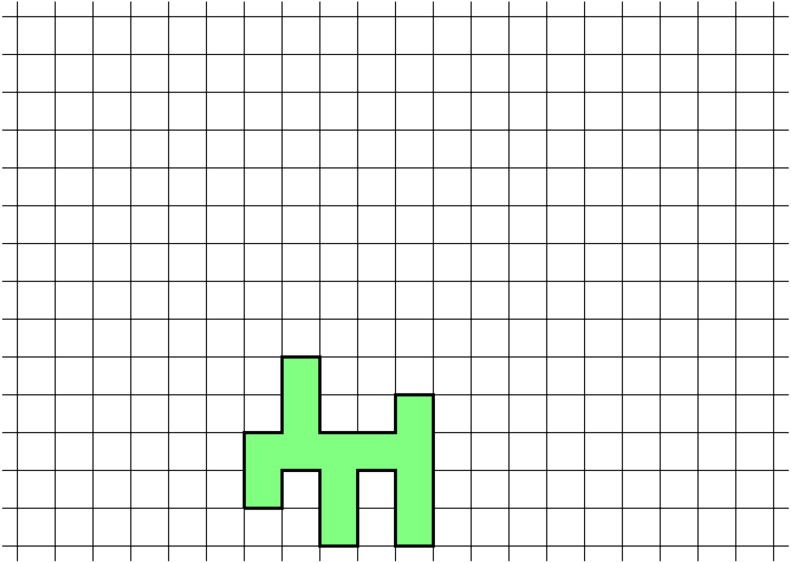
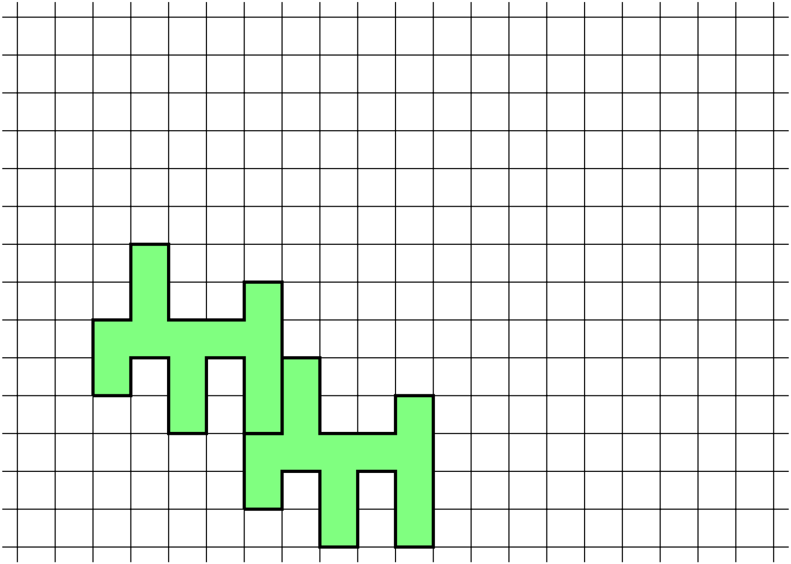
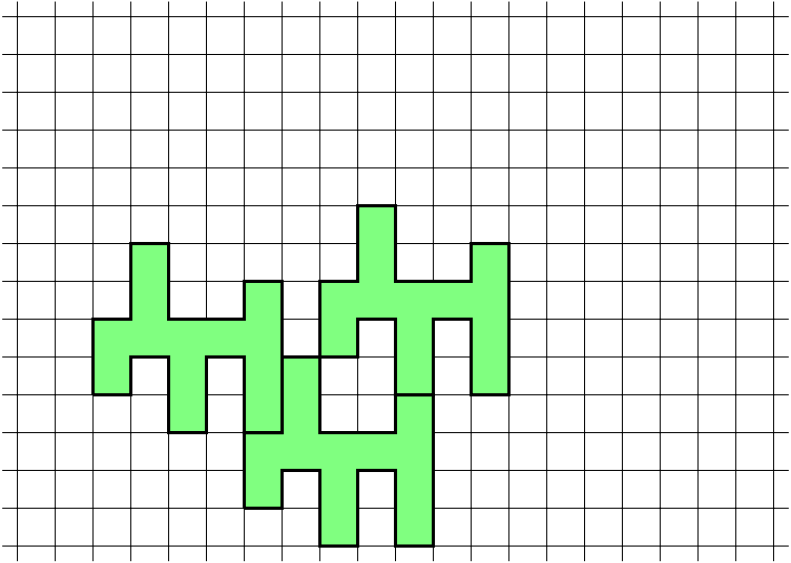
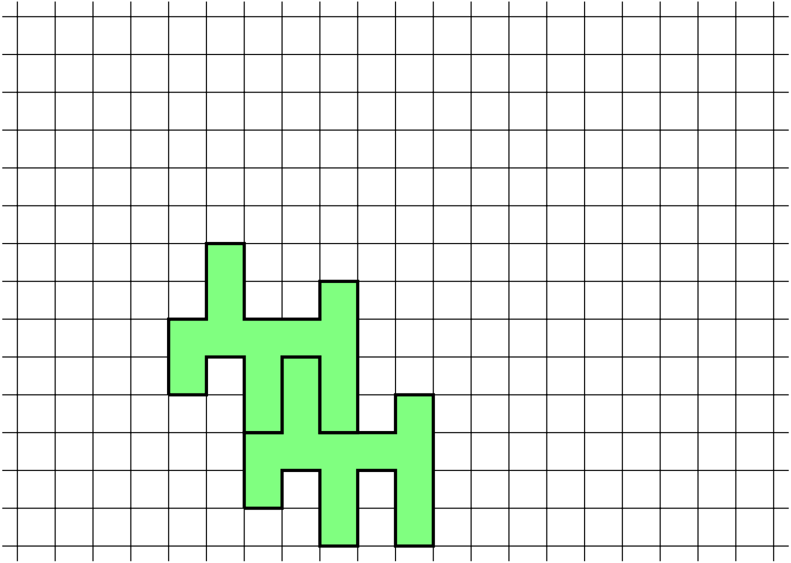
Problem

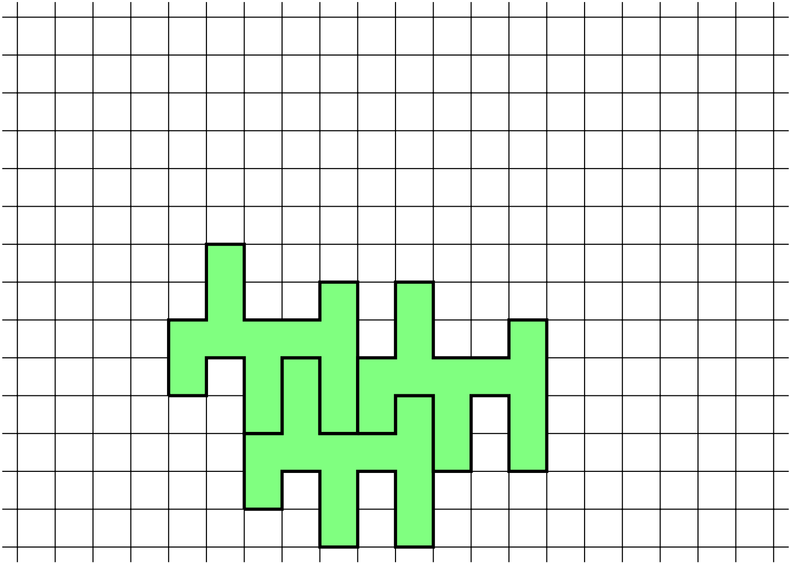
Does a given polyomino P tile the plane?

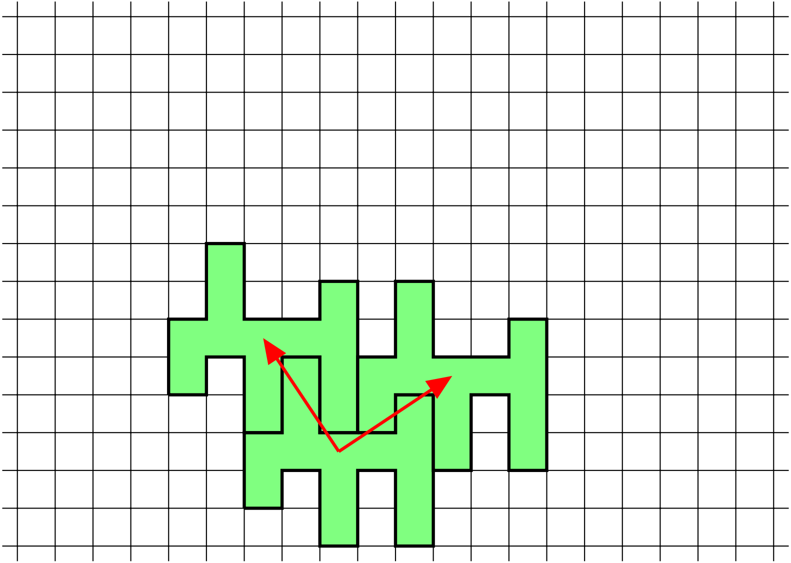


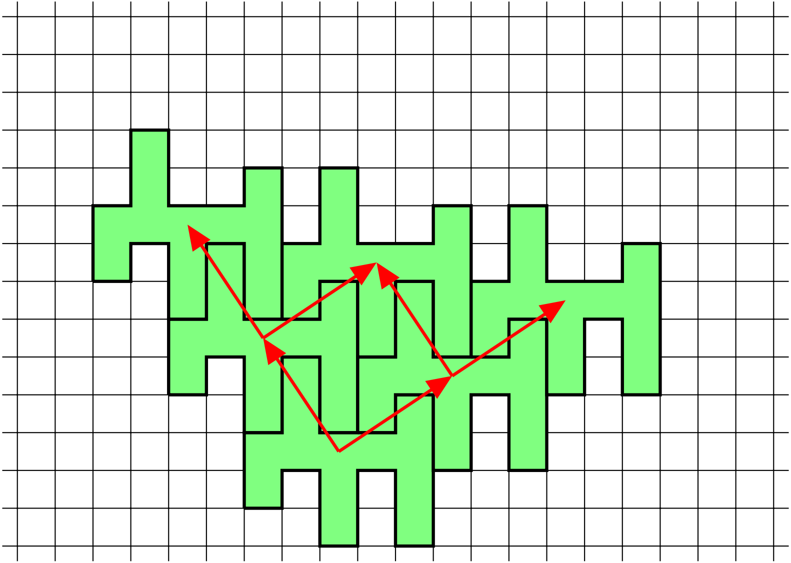


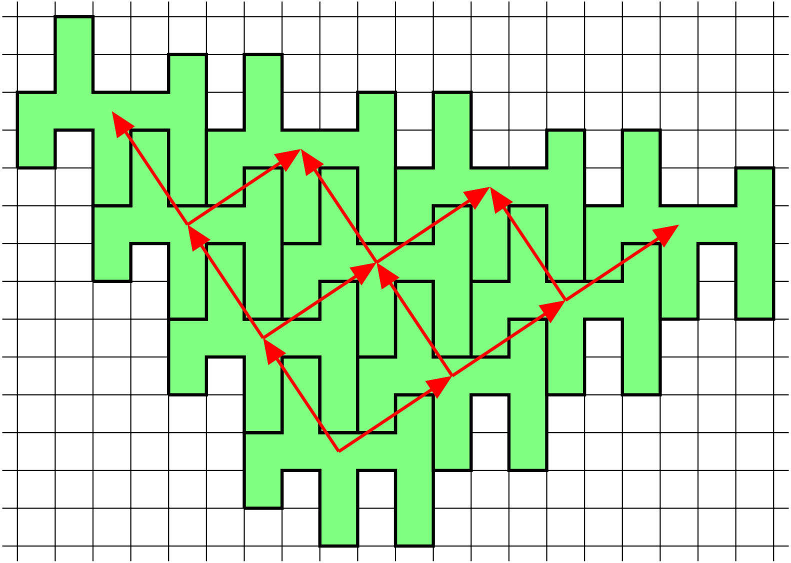


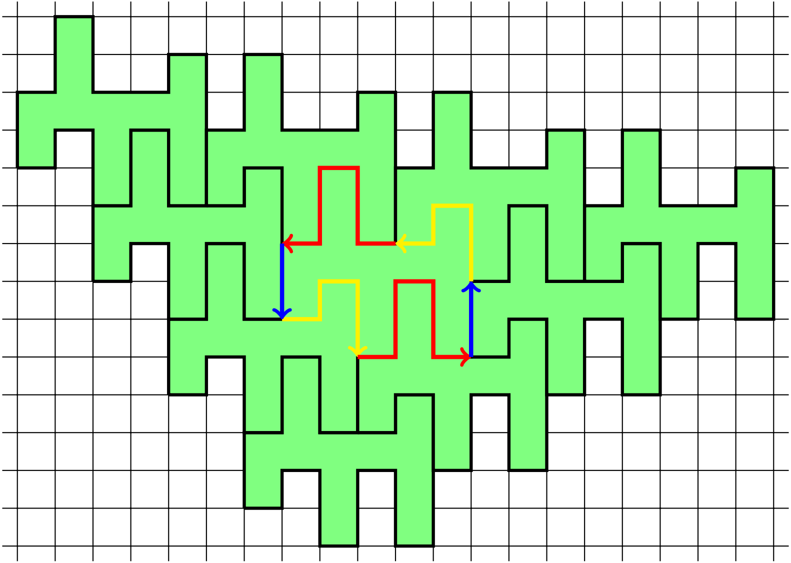








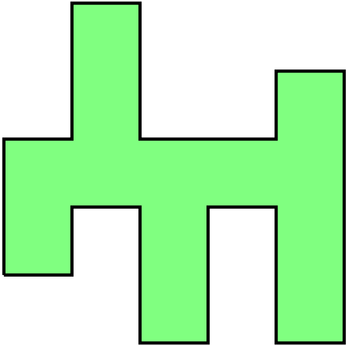




Freeman Chain Code

$$\Sigma = \{a, \overline{a}, b, \overline{b}\}$$

$a \rightarrow$	$b \uparrow$
$\overline{a} \leftarrow$	$\overline{b} \downarrow$



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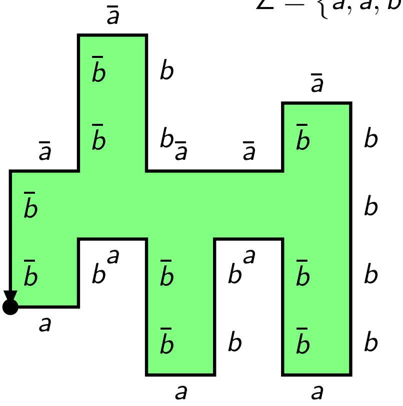

$$w = aba\overline{bb}abba\overline{bb}abbbb\bar{a}\bar{b}\overline{aa}bb\bar{a}\overline{bb}\bar{a}\overline{bb}$$

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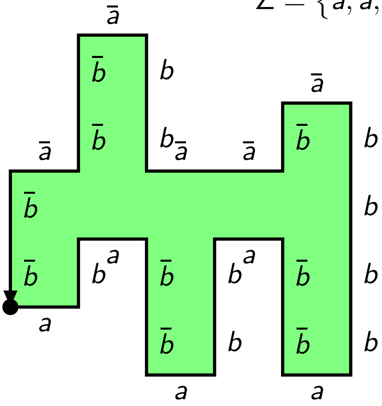

$a \rightarrow$	$b \uparrow$
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Any conjugate w' of w codes the same polyomino.

w and w' are conjugate if there exist $u, v \in \Sigma^*$ such that $w = uv$ and $w' = vu$.

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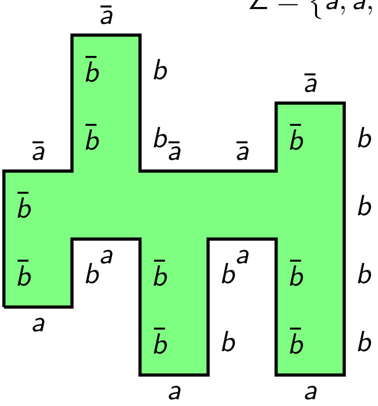

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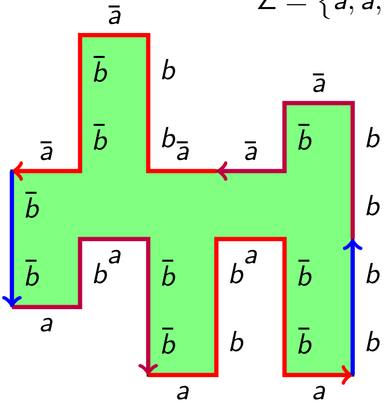

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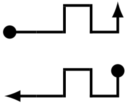
$$w \equiv aba\overline{bb}abba\overline{bb}abbbb\overline{a}\overline{b}\overline{aa}bb\overline{a}\overline{bb}\overline{a}\overline{bb}$$

Beauquier and Nivat (1991)

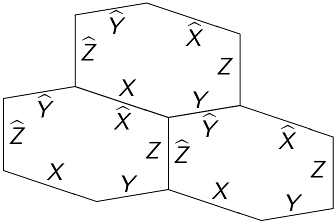
Characterization: A polyomino P tiles the plane if and only if there exist $X, Y, Z \in \Sigma^*$ such that $w \equiv XYZ\widehat{X}\widehat{Y}\widehat{Z}$.

$X = a\ a\ b\ a\ \overline{b}\ a\ b$

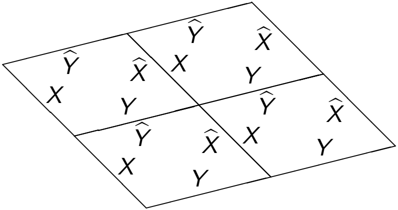
$\widehat{X} = \overline{b}\ \overline{a}\ b\ \overline{a}\ \overline{b}\ \overline{a}\ \overline{a}$



hexagon tiles

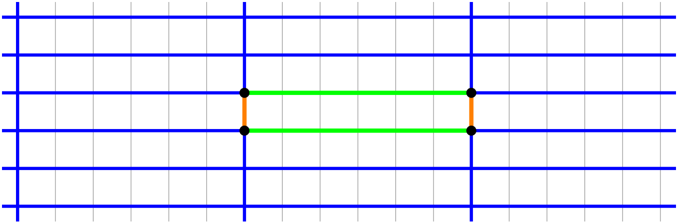


square tiles



Maurits Cornelis Escher (1898-1972). Hexagonal tiling. Square tiling.

There are polyominoes admitting many hexagon tilings:



A $1 \times n$ rectangle tiles the plane as an hexagon in $n-1$ ways and as a square in only 1 way.

Hexagonal Tilings

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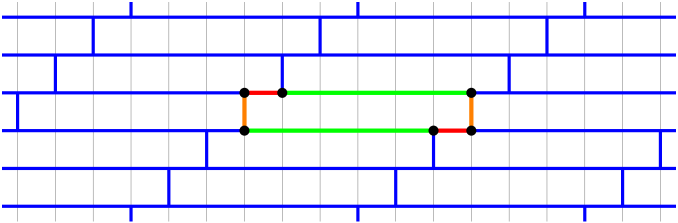

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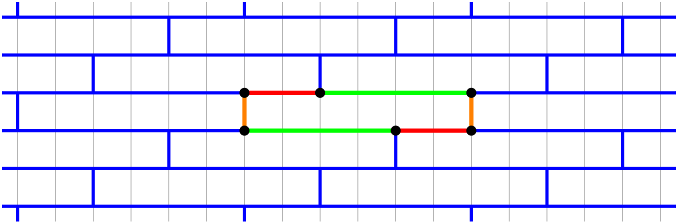

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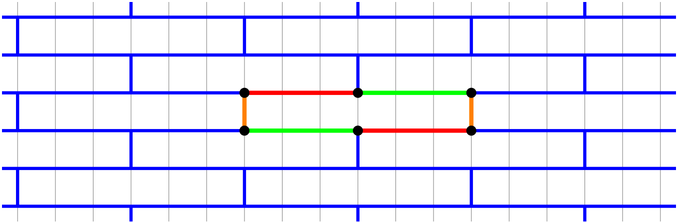

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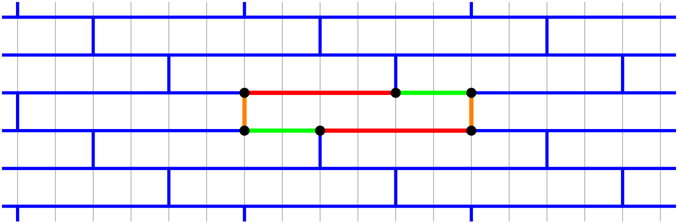

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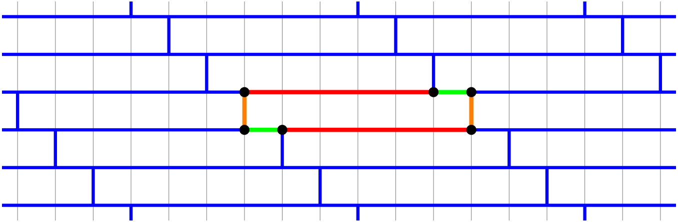

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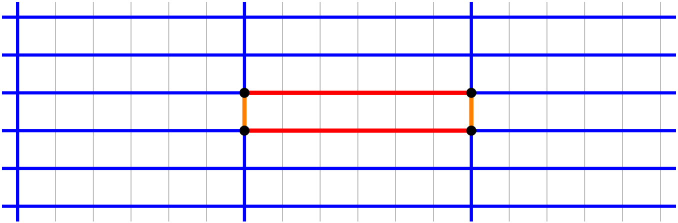

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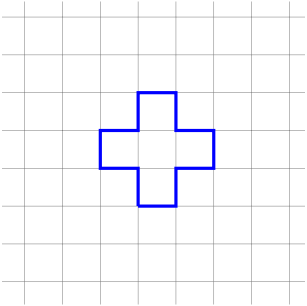
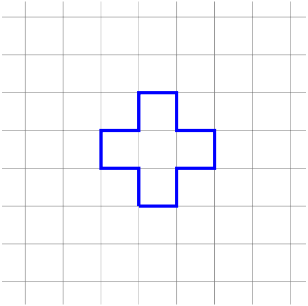
Square Tilings

The pentamino has two distinct square factorizations:





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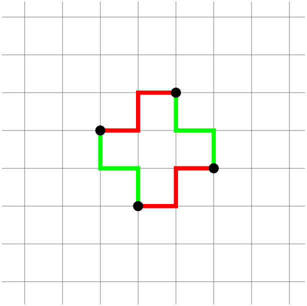

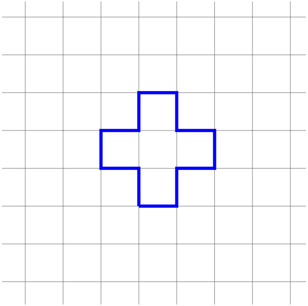

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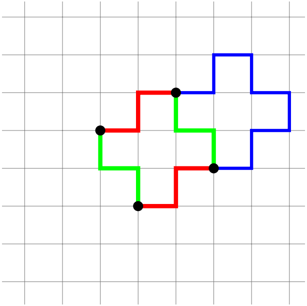

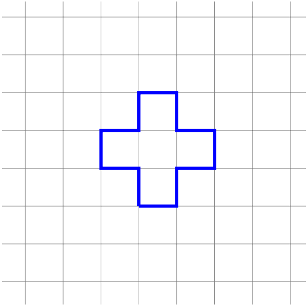

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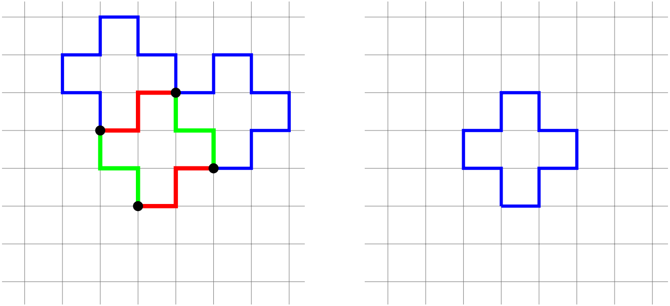

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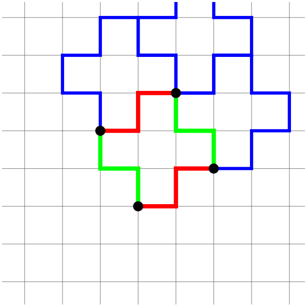

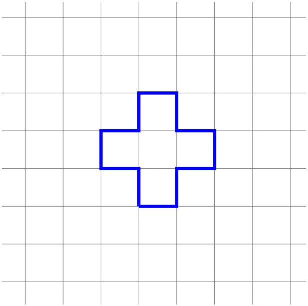

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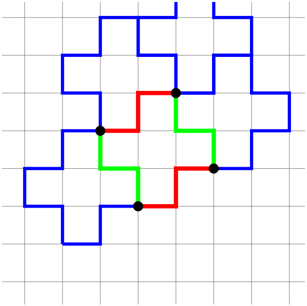

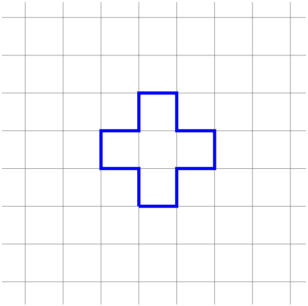

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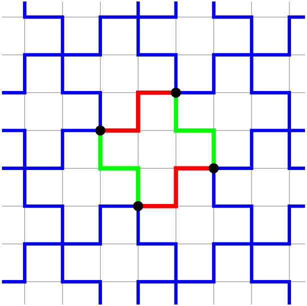

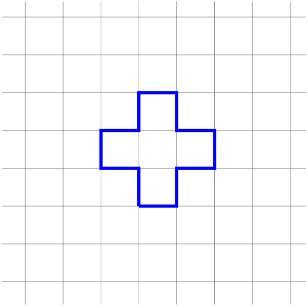

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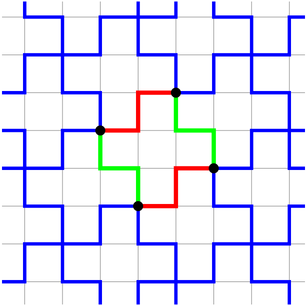

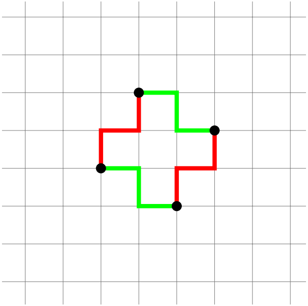

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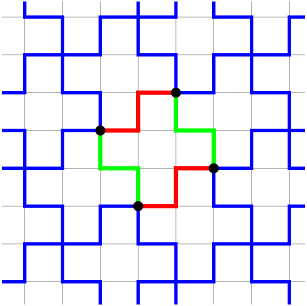

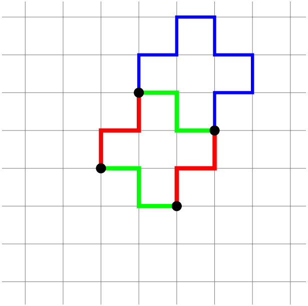

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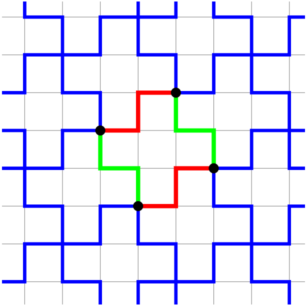

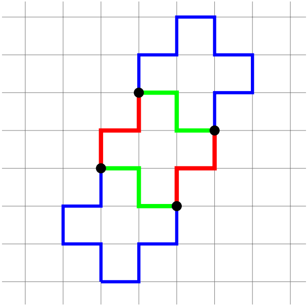

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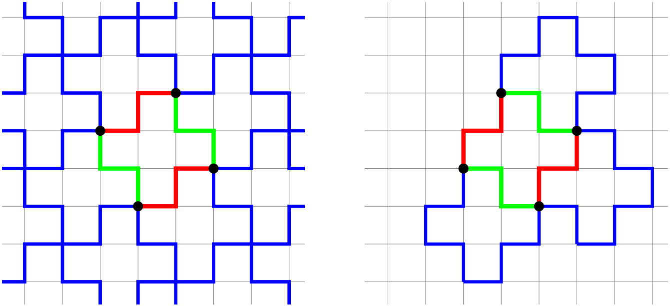

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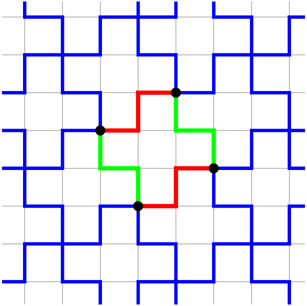

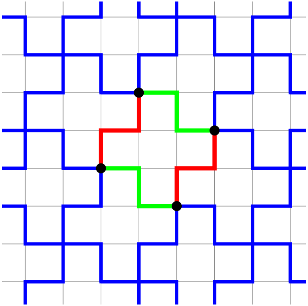

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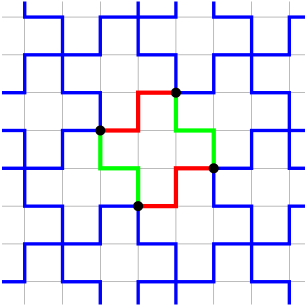
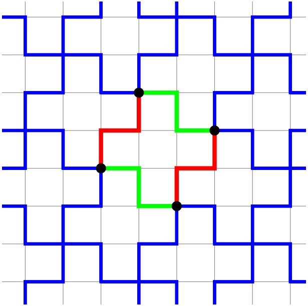




Conjecture (Brlek, Dulucq, Fédou, Provençal 2007)

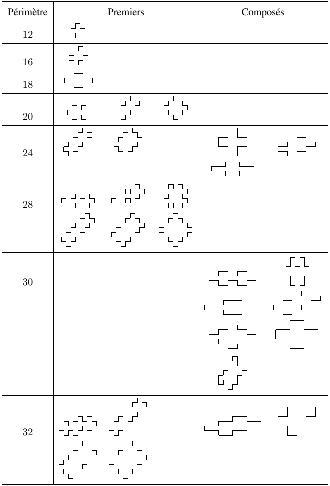
A tile has at most 2 square factorizations.

Definition

A double square is a tile having two distinct square factorizations.

Table of the first double squares (Provençal's Thesis, p.96):

96



Périmètre	Premiers	Composés
12		
16		
18		
20		
24		
28		
30		
32		

Tableau 4.1 Les doubles pseudo-carrés de périmètre inférieur ou égal à 32.

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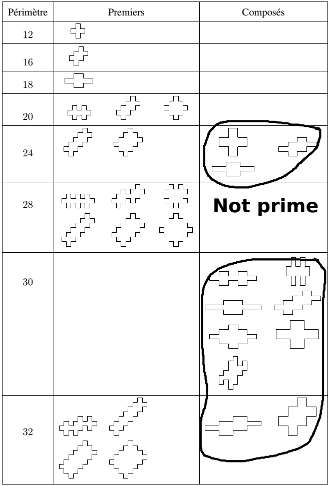
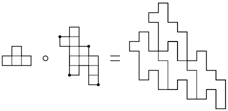


Tableau 4.1 Les doubles pseudo-carrés de périmètre inférieur ou égal à 32.

Prime Tiles

Let P be a polyomino and S be a square tile. Then the composition $P \circ S$ is the polyomino defined by replacing each unit cell of P by S.



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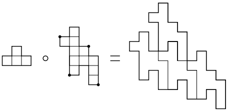

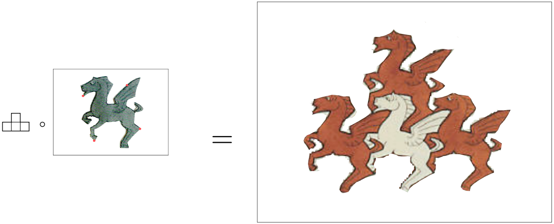

Note: This is not commutative.

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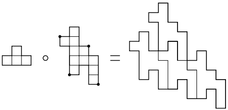
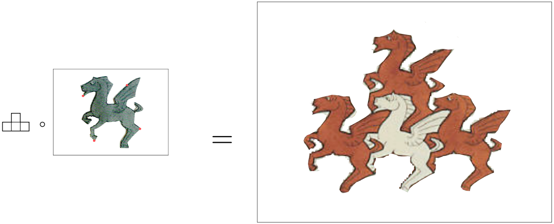




Note: This is not commutative.

Definition

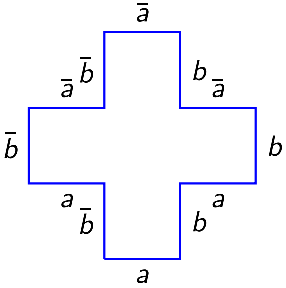
A polyomino Q is prime if $Q = P \circ S$ implies that P or S is the 1×1 unit square.

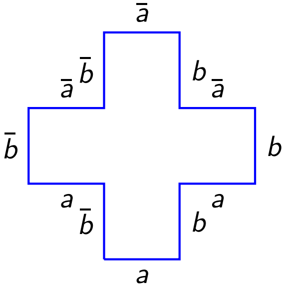
Prime Double Squares

Conjecture (X. Provençal and L. Vuillon, 2008)

If $XY\widehat{X}\widehat{Y}$ describes the contour of a prime double square, then both X and Y are palindromes.

Note: a palindrome is a word that reads the same forward as it does backward.





Prime Double Squares

Conjecture (X. Provençal and L. Vuillon, 2008)

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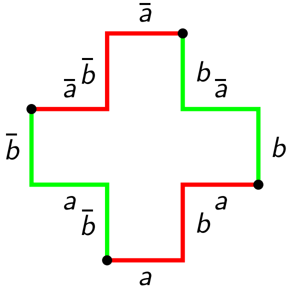

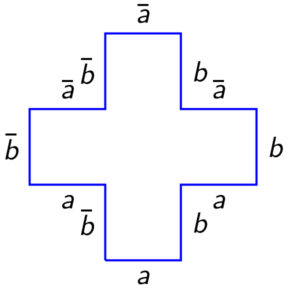

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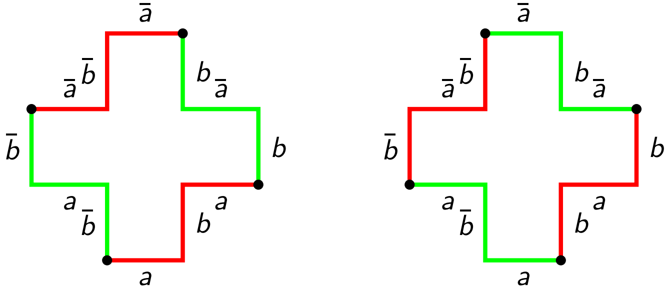

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Table of the first double squares (Provençal's Thesis, p.96):

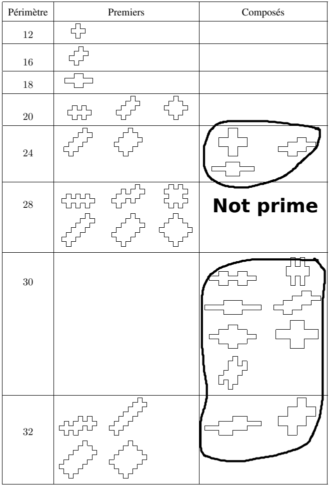

Tableau 4.1 Les doubles pseudo-carrés de périmètre inférieur ou égal à 32.

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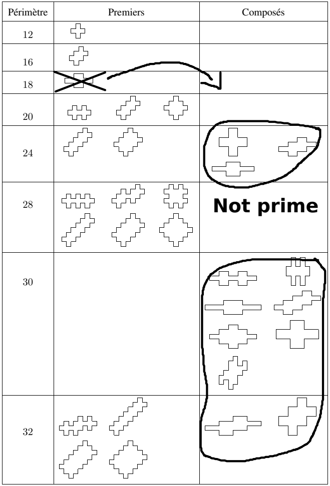
96



Périmètre	Premiers	Composés
12		
16		
18		
20		
24		
28		
30		
32		

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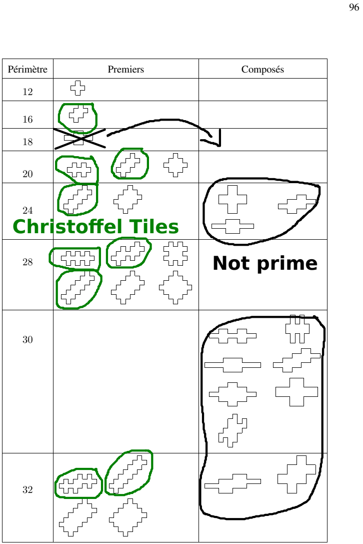

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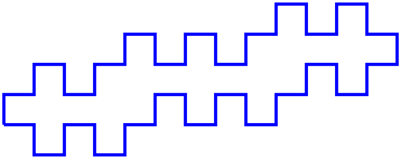
Christoffel Tiles

There are prime double squares like the one below



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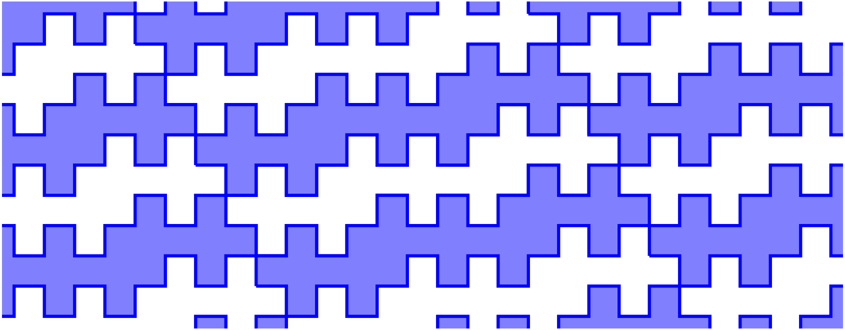

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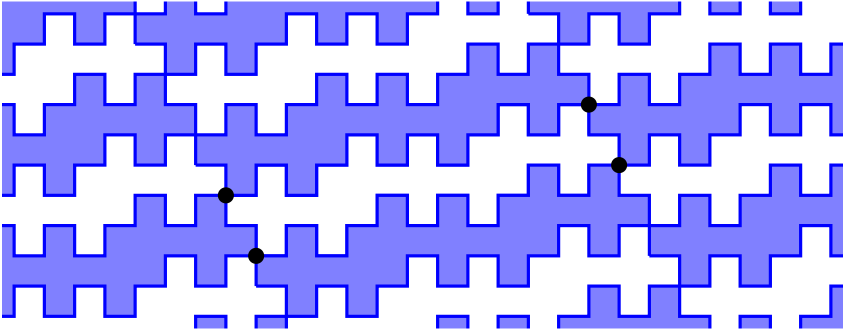

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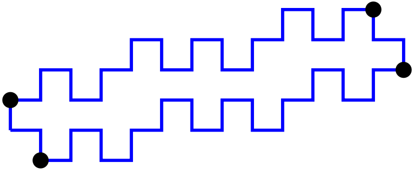

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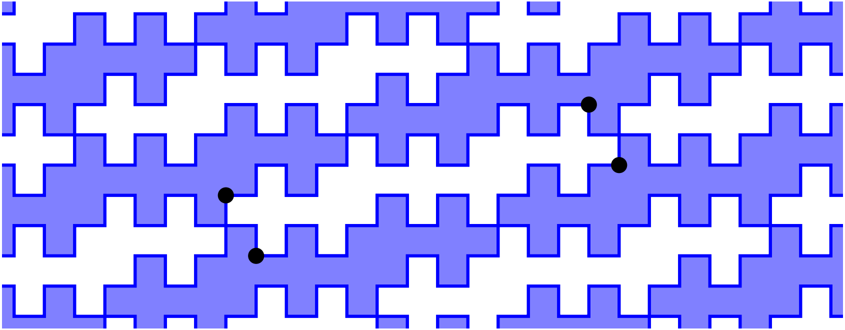

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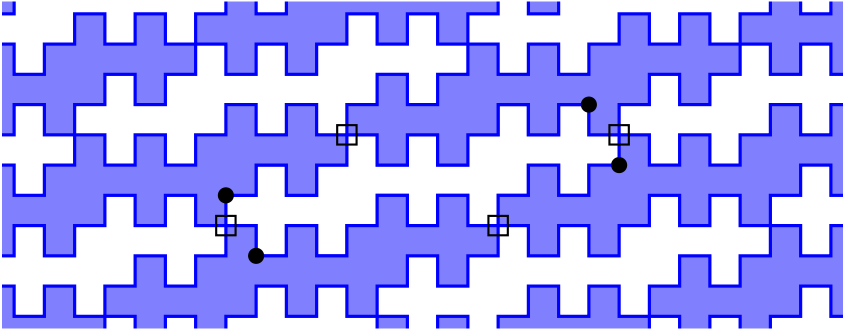

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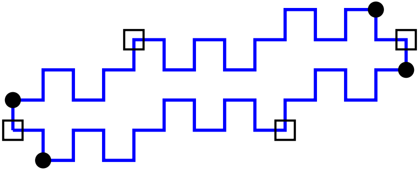

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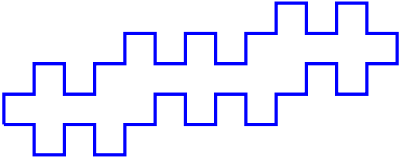
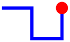
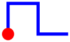
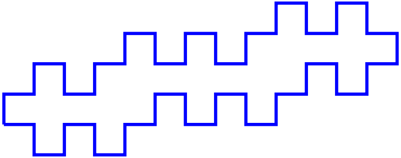


There are prime double squares like the one below that may be factorized in , ,  and  :



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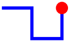
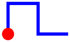
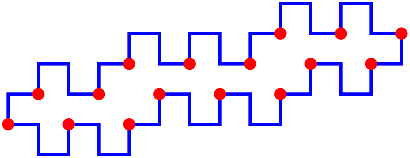
There are prime double squares like the one below that may be factorized in , , and  :



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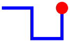
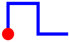
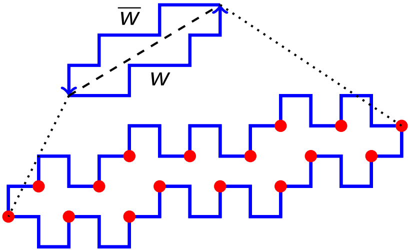
There are prime double squares like the one below that may be factorized in , ,  and  :



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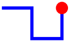
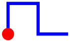
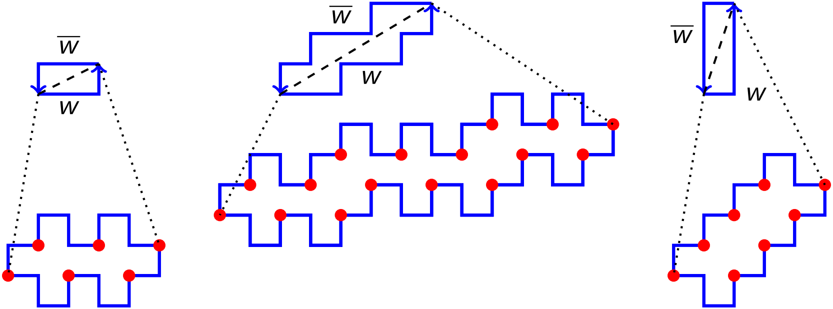
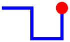
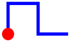
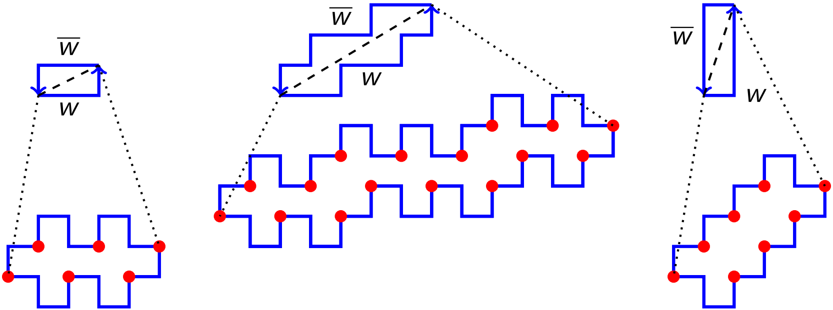
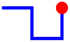
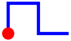
There are prime double squares like the one below that may be factorized in , ,  and  :



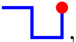
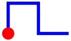
There are prime double squares like the one below that may be factorized in , ,  and  :



Those three can be obtained from smaller words $w\overline{w}$ via the morphism

$\lambda : a \mapsto$ , $b \mapsto$ , $\bar{a} \mapsto$ , $\bar{b} \mapsto$  .

Christoffel Tiles

$\lambda : a \mapsto$ , $b \mapsto$ , $\bar{a} \mapsto$ , $\bar{b} \mapsto$ .

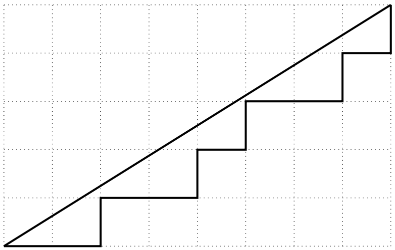
Theorem (Blondin Massé, Brlek, Garon, L.)

Let $w = apb$ where a and b are letters.

(i) *If p is a palindrome, then $\lambda(w\overline{w})$ is a square tile.*

(ii) *$\lambda(w\overline{w})$ is a double square if and only if w is a Christoffel word.*

Christoffel words are discretization of finite segments.



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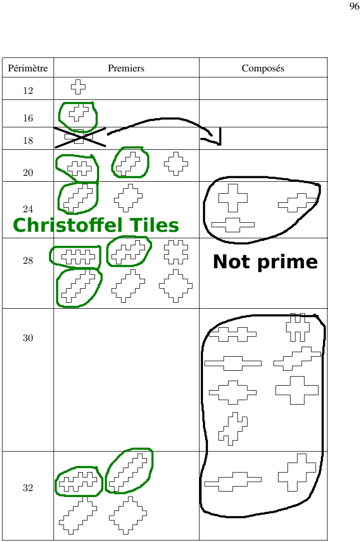

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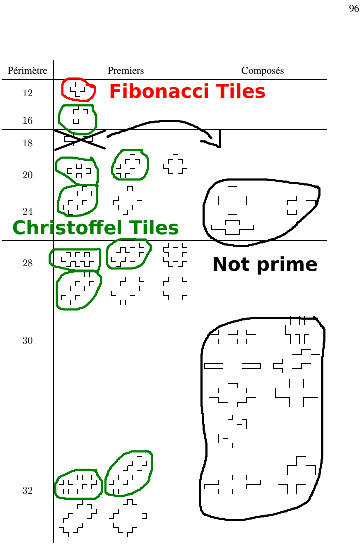


Tableau 4.1 Les doubles pseudo-carrés de périmètre inférieur ou égal à 32.

Fibonacci Tiles

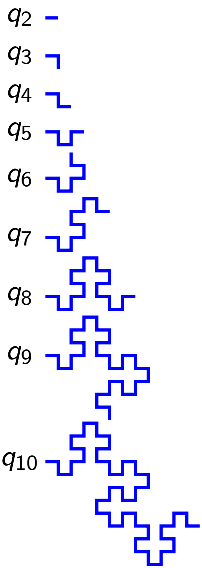
We define a sequence in $\{\mathrm{R},\mathrm{L}\}^*$ by $q_0 = \varepsilon$, $q_1 = \mathrm{R}$ and

$$q_n = \begin{cases} q_{n-1}q_{n-2} & \text{if } n \equiv 2 \mod 3, \\ q_{n-1}\overline{q_{n-2}} & \text{if } n \equiv 0,1 \mod 3. \end{cases}$$

The first terms are

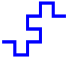
$$\begin{array}{lllllllll} q_0 & = & \varepsilon & q_3 & = & \mathrm{RL} & q_6 & = & \mathrm{RLLRLLRR} \\ q_1 & = & \mathrm{R} & q_4 & = & \mathrm{RLL} & q_7 & = & \mathrm{RLLRLLRRLRRLR} \\ q_2 & = & \mathrm{R} & q_5 & = & \mathrm{RLLRL} & q_8 & = & \mathrm{RLLRLLRRLRRLRRLLRLLRR} \end{array}$$

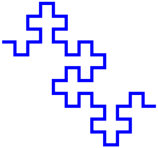
Fibonacci Tiles



Fibonacci Tiles

q_4 

q_7 

q_{10} 

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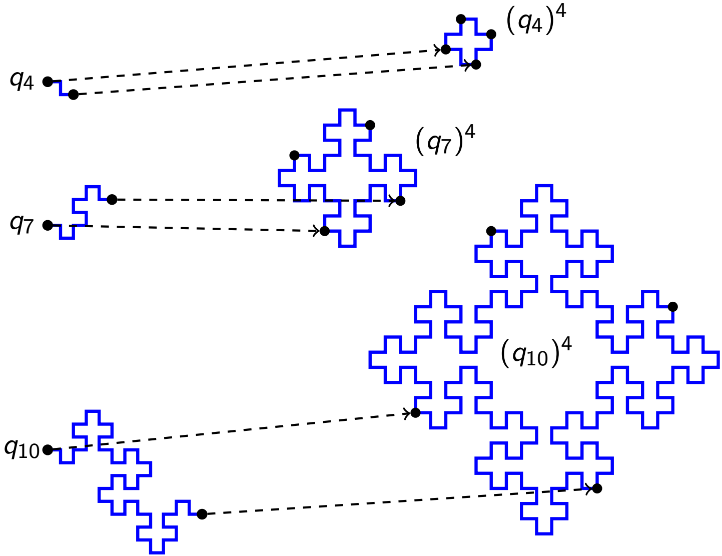

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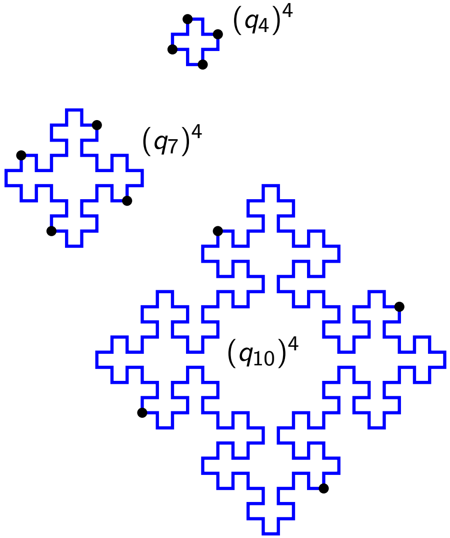

Lemma (Blondin Massé, Brlek, L., Mendès France)

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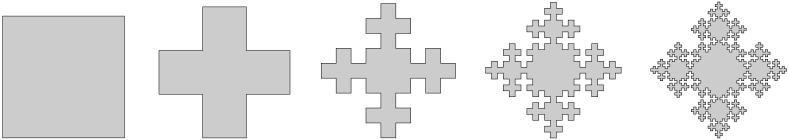

Figure: Fibonacci tiles of order $n = 0, 1, 2, 3, 4$.

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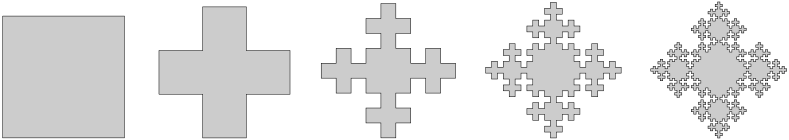


Figure: Fibonacci tiles of order $n = 0, 1, 2, 3, 4$.

Theorem (Blondin Massé, Brlek, Garon, L.)

(i) *Fibonacci tiles of order $n \geq 1$ are double squares.*

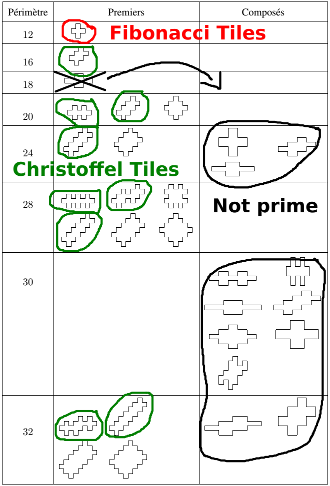
(ii) *If $AB\hat{A}\hat{B}$ is a BN-factorisation of a Fibonacci tile, then A and B are palindromes.* □

Definition

A double square is a tile having two distinct square factorizations.

Table of the first double squares (Provençal's Thesis, p.96):

96



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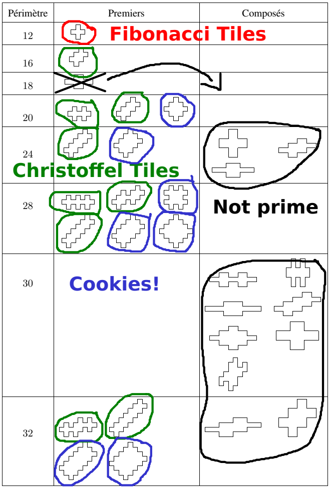
Cookies are crenelated versions of Fibonacci tiles.

96



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Problem

Does the Christoffel tiles and (generalized) Fibonacci tiles describe all the prime double square tiles?

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No!!!!

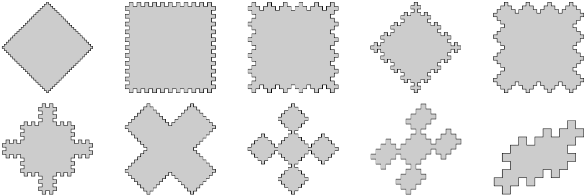


Figure: Some double squares not in the Christoffel tiles nor in the Fibonacci tiles families.

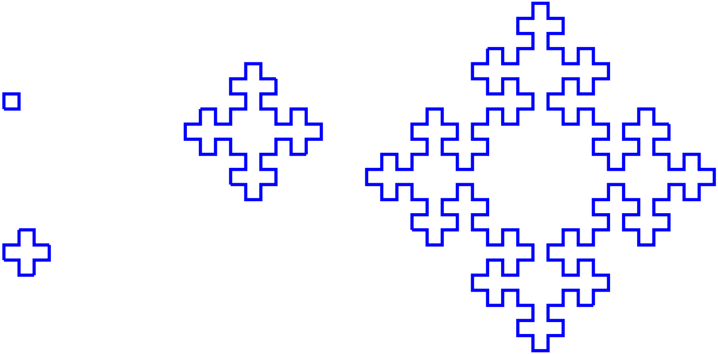
Useful Software

This research was driven by computer exploration using the open-source mathematical software Sage [1] and its algebraic combinatorics features developed by the Sage-Combinat community [2], and in particular, F. Saliola, A. Bergeron and S. Labbé.
The pictures have been produced using Sage, pgf/tikz and Xournal.

W. A. Stein et al., *Sage Mathematics Software (Version 4.1.1)*, The Sage Development Team, 2009, http://www.sagemath.org.

The Sage-Combinat community, Sage-Combinat: enhancing Sage as a toolbox for computer exploration in algebraic combinatorics, http://combinat.sagemath.org, 2009.

A new relation between Fibonacci and Pell numbers



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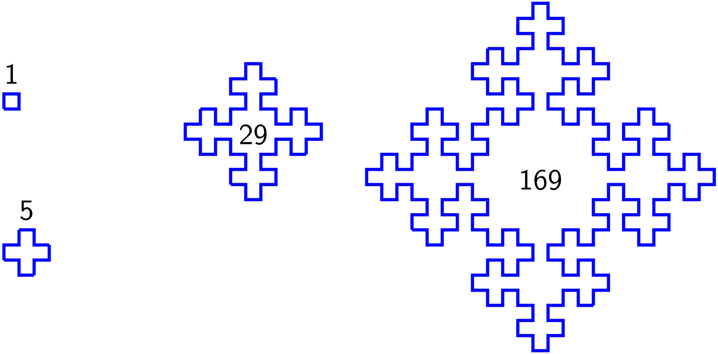

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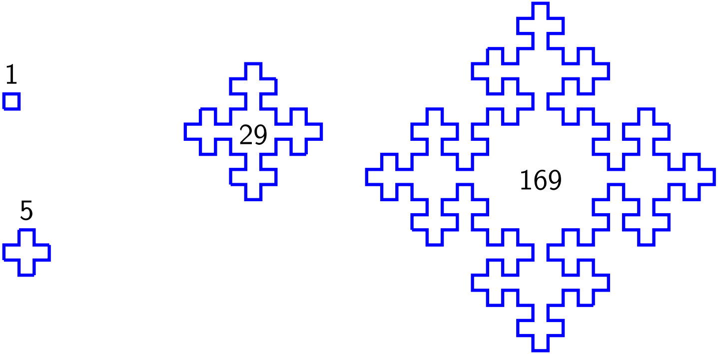

This is the subsequence of odd index Pell numbers

$$0, 1, 2, 5, 12, 29, 70, 169, 408, 985, 2378, 5741, 13860, 33461, 80782, \ldots$$

defined by $P_0 = 0$, $P_1 = 1$, $P_n = 2P_{n-1} + P_{n-2}$, for $n > 1$,

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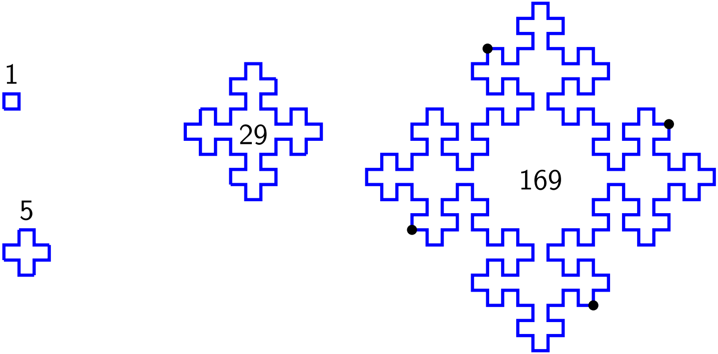

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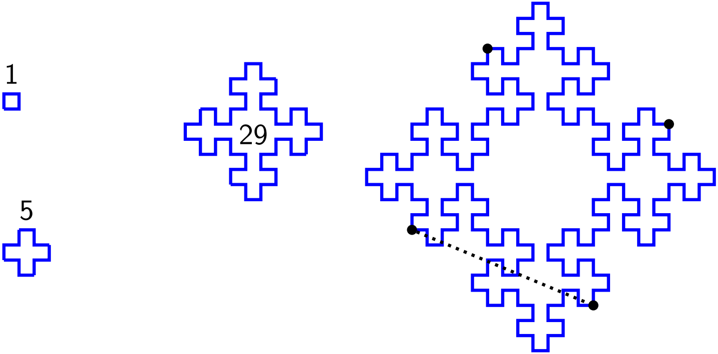

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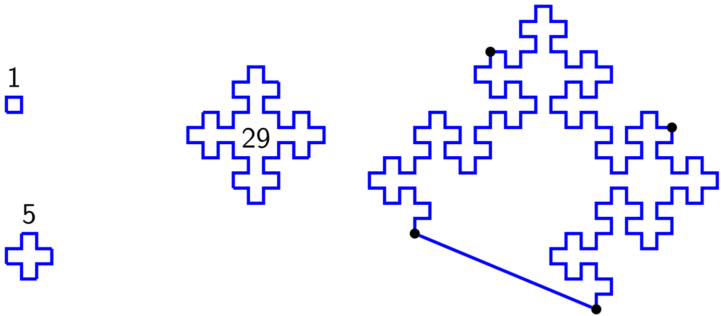

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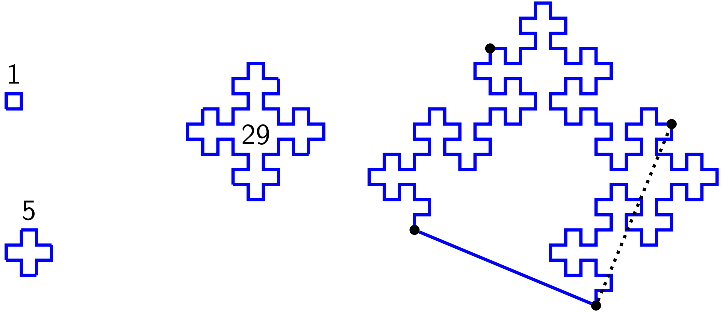

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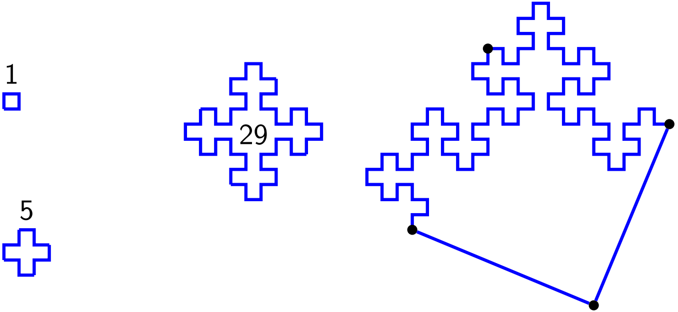

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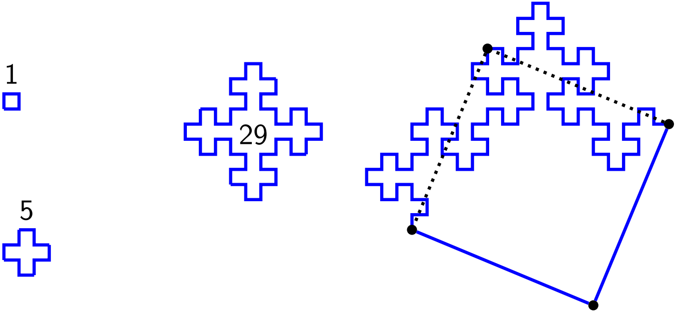

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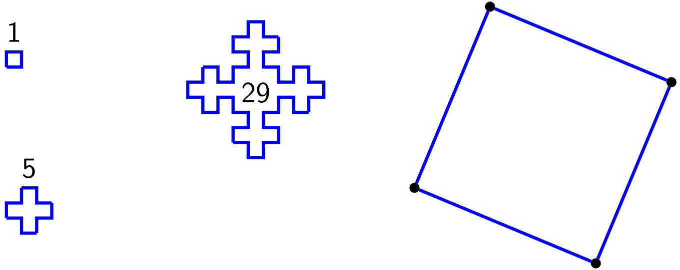

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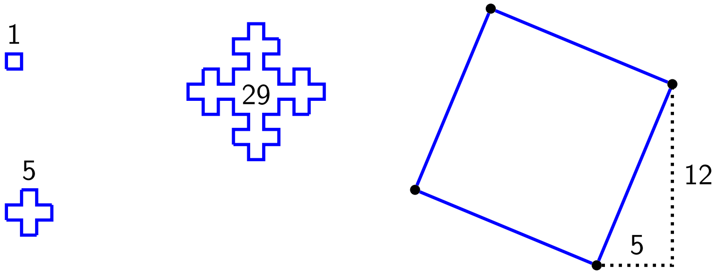

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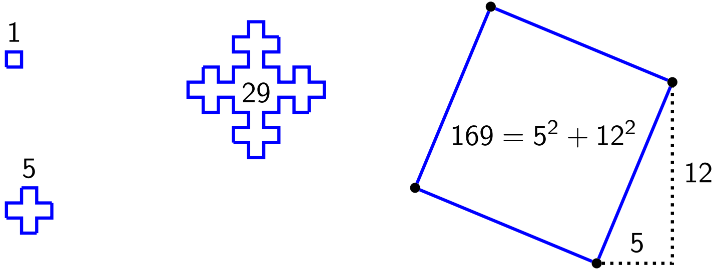

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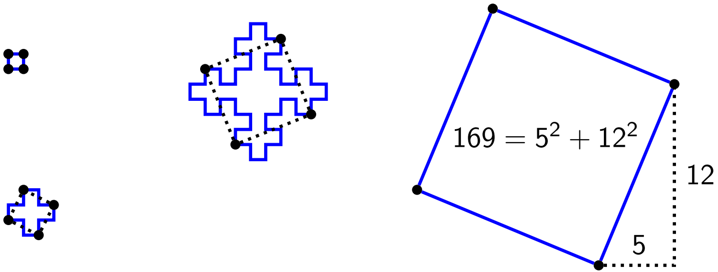

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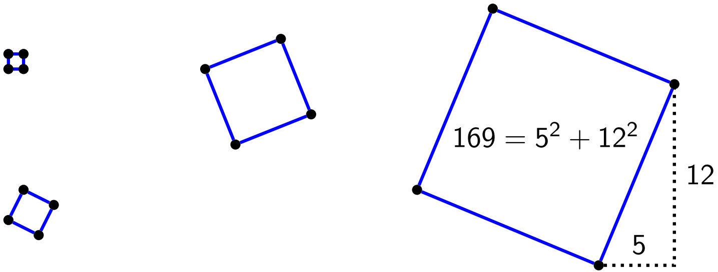

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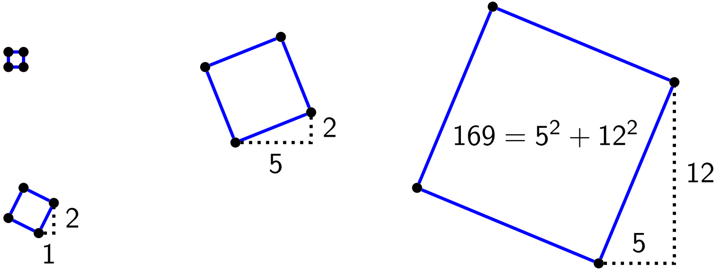

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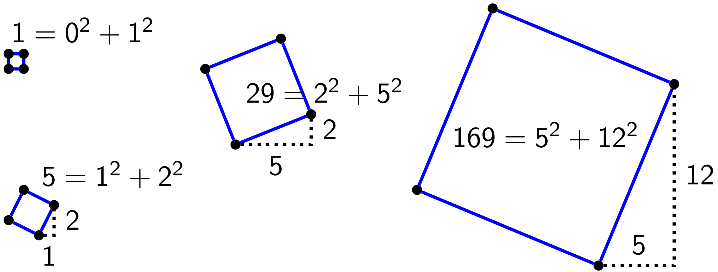

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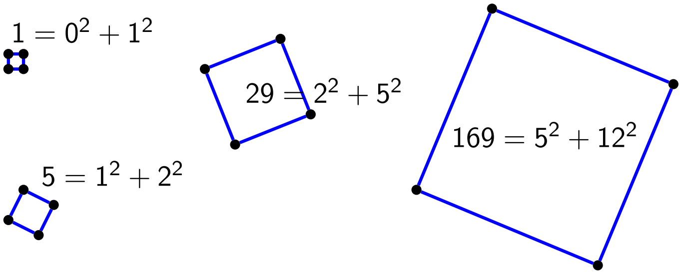

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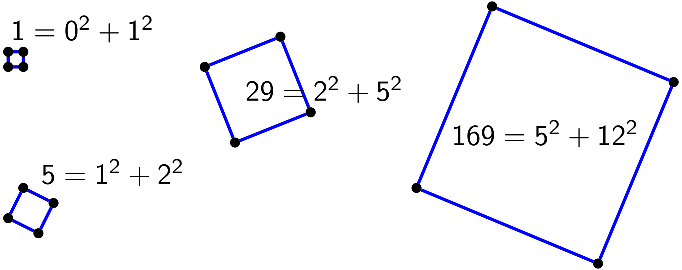

This is the subsequence of odd index Pell numbers

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defined by $P_0 = 0$, $P_1 = 1$, $P_n = 2P_{n-1} + P_{n-2}$, for $n > 1$,
which also satisfies $P_n^2 + P_{n+1}^2 = P_{2n+1}$ for all $n \geq 0$ [Putnam 1999].