A note on the critical exponent of generalized Thue-Morse words

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The Thue-Morse word **m** is obtained as the fixpoint of the morphism

μ :	Σ^*	\rightarrow	Σ*
	а	\mapsto	ab
	Ь	\mapsto	

where $\Sigma = \{a, b\}$ is a binary alphabet, i.e.

 $\mu(\mathbf{m}) = \mathbf{m} = abbabaabbaabbaabba \cdots$

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The Thue-Morse word can also be obtained from the numeric sequence

$$\left(s_2(n) \mod 2\right)_{n \ge 0} = 01101001\cdots$$

where $s_2(n)$ denotes the sum of the digits of *n* written in base 2. Allouche and Shallit [2000] generalized this sequence :

$$\mathbf{t}_{b,m} = \left(s_b(n) \bmod m\right)_{n \ge 0}$$

The integer power of a word w is defined by

$$w^n = \begin{cases} \varepsilon & \text{if } n = 0\\ ww^{n-1} & \text{if } n > 0 \end{cases}.$$

Definition

Let r be a rational such that $r|w| \in \mathbb{N}$. The rational power of a word w is defined by

$$w^r = w^{\lfloor r \rfloor} p,$$

where p is the prefix of w of length $(r - \lfloor r \rfloor)|w|$.

Consider the word *abb*. Then

(abb) ⁰	=	ε
$(abb)^1$	=	abb
(abb)²	=	abbabb
(<i>abb</i>) ^{2/3}	=	ab
(abb) ^{8/3}	=	abbabbab
(abb) ^{5/2}	is	not defined

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$$x = baaa {abbabbabbab} (abb)^{11/3} (abb)^{8/3}$$

Rational powers of abb appear in x

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DefinitionThe index of a word w in another word x is given by the number $Index(w) = sup\{r \in \mathbb{Q} \mid w^r \text{ is a factor of } x\}$

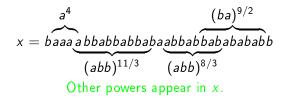
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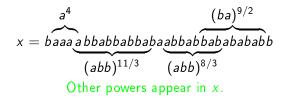
The index of *abb* in x is 11/3.

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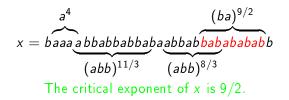
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Definition

The critical exponent E(x) of a word x is the greatest index realized by its factor, i.e.

$$E(x) = \sup\{ \ln \det(w) \mid w \neq \varepsilon \text{ is a factor of } x \}.$$



Definition

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- Mignosi and Pirillo [1992] showed that the Fibonacci word **f** has critical exponent $E(\mathbf{f}) = 2 + \varphi$, where φ is the golden ratio.
- Dalia Krieger [2007] studied the critical exponent of fixpoint of non-erasing morphisms. In particular, she showed that for uniform morphisms, it is either infinite or rational.
- Allouche and Shallit [2000] showed that $E(t_{b,m}) = 2$ for $b \le m$.

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- Allouche and Shallit [2000] showed that E(t_{b,m}) = 2 for b ≤ m.

Our contribution is to compute $E(\mathbf{t}_{b,m})$ when b > m.

Generalized Thue-Morse word

Definition

Let $b \geq 2$, $m \geq 1$ be integers, Σ an alphabet of m letters, $\sigma: \Sigma \to \Sigma$ a cyclic permutation and $\mu: \Sigma^* \to \Sigma^*$ the morphism given by

$$\mu(\alpha) = \sigma^{0}(\alpha)\sigma^{1}(\alpha)\sigma^{2}(\alpha)\cdots\sigma^{b-1}(\alpha).$$

The generalized Thue-Morse word t, beginning with $\bar{\alpha} \in \Sigma$, is the infinite word given by $\mathbf{t} = \mu^{\omega}(\bar{\alpha})$.

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Example

With $\Sigma = \{a, b\}$, $\sigma : a \mapsto b \mapsto a$ and $\overline{\alpha} = a$, we get $\mu(a) = ab$, $\mu(b) = ba$ and

 $\mathbf{t}=\mu^{\omega}(\mathbf{a})=\mathbf{a}bbabaabbaabbaabba \cdots$

which is the original Thue-Morse word **m**.

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Remark

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The n-th letter of t is \sigma^{s_b(n)}(\bar{\alpha}).
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Remark

The *n*-th letter of t is $\sigma^{s_b(n)}(\bar{\alpha})$.

Since 5 is written as 101 in base 2, the 5^{th} letter of the Thue-Morse word **m** is

$$\sigma^{s_2(5)}(a) = \sigma^{1+0+1}(a) = \sigma^2(a) = a,$$

as it should be :

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With

$$egin{aligned} \Sigma &= \mathbb{Z}_m \ \sigma(i) &= (i+1) egin{aligned} & \ m & \ ar{lpha} &= 0, \end{aligned}$$

the previous remark implies that the n-th letter of t is

$$\sigma^{s_b(n)}(0) = s_b(n) \mod m,$$

which is the word $t_{b,m}$ introduced by Allouche and Shallit [2000].

Let b = 5, m = 3, $\Sigma = \{ \triangle, \diamondsuit, \heartsuit \}$ and $\sigma : \triangle \mapsto \diamondsuit \mapsto \oslash \mapsto \triangle$. This gives the following morphism

$$\begin{array}{ccccc} \mu & : & \bigtriangleup & \mapsto & \bigtriangleup \heartsuit \bigtriangleup \diamondsuit & \\ & & & & & \diamondsuit & \diamondsuit & \circlearrowright \bigtriangleup \oslash & \\ & & & & & \heartsuit \bigtriangleup \oslash \bigtriangleup & \\ \end{array}$$

By fixing $\bar{\alpha} = \Diamond$, we obtain the following generalized Thue-Morse word $\mathbf{t}_{\clubsuit} = \mu^{\omega}(\Diamond) = \diamondsuit$

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Example

Let b = 7, m = 3, $\Sigma = \{0, 1, 2\}$ and $\sigma : 0 \mapsto 1 \mapsto 2 \mapsto 0$. This gives the following morphism

μ	:	0	\mapsto	0120120
		1	\mapsto	1201201
		2	\mapsto	2012012

and the following periodic fixpoint

$$\mathbf{t}=\mu^{\omega}(\mathbf{0}) = \mathbf{0}$$

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= $(\mathbf{0}12)^{\omega}$

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= (012)^{\overline{\overline{0}}} = (012)^\overline{\overline{0}}}

Lemma [Morton and Mourant, 1991]

t is periodic if and only if $m \mid (b-1)$.

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Assuming that t is aperiodic, i.e. $m \nmid b - 1$.

Theorem

The critical exponent of t is given by

$$E(\mathbf{t}) = \begin{cases} 2b/m & \text{if } b > m \\ 2 & \text{if } b \le m \end{cases}$$

We denote

$$S^p_{x,y} = \{s \in \mathbb{N}^+ \mid p = sx + ty, t \in \mathbb{Z}\}$$

the arithmetical progression associated to p, x and y.

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We denote

$$S_{x,y}^{p} = \{s \in \mathbb{N}^{+} \mid p = sx + ty, t \in \mathbb{Z}\}$$

the arithmetical progression associated to p, x and y. Then, we consider the three following sets :

$$\begin{array}{lll} A &=& \left\{ \, k b^{\,q} - b \,\mid\, k \in \mathbb{N}^{+}, b \nmid k \text{ and } q \in S^{m}_{b-1,m} \right\}, \\ B_{N} &=& \left\{ \, k b^{\,q} - N \,\mid\, k \in \mathbb{N}^{+}, b \nmid k \text{ and } q \in S^{N}_{b-1,m} \right\}, \\ C &=& (8 \cdot B_{1} + 3) \cup (8 \cdot B_{1} + 7). \end{array}$$

Assuming that t is aperiodic, i.e. $m \nmid b - 1$.

Theorem

Critical factors of length Nb^i such that $b \nmid N$ occur in t at the following set of positions

$$\left\{ egin{array}{ll} b^iA & ext{if }b>m,\ b^iB_N & ext{if }b\leq m ext{ and }(b,m)
eq(2,2),\ b^i(B_1\cup C) & ext{if }b=m=2. \end{array}
ight.$$

Recall the generalized Thue-Morse word that has already been defined on $\Sigma=\{\triangle,\diamondsuit,\heartsuit\}$

$$\mathbf{t}_{\clubsuit} = \Diamond \heartsuit \triangle \Diamond \heartsuit \heartsuit \triangle \Diamond \heartsuit \triangle \Diamond \oslash \triangle \Diamond \oslash \triangle \Diamond \heartsuit \triangle \Diamond \heartsuit \cdots$$

From the first Theorem, $E(\mathbf{t}_{\bullet}) = \frac{2b}{m} = \frac{10}{3}$.

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From the first Theorem, $E(\mathbf{t}_{\clubsuit}) = \frac{2b}{m} = \frac{10}{3}$. From the second Theorem, we compute

$$S_{5-1,3}^3 = \{3, 6, 9, 12, 15, \ldots\}$$

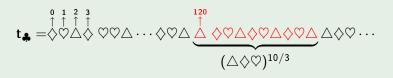
and obtain the set of positions

 $A = \{120, 245, 370, 495, 745, \ldots\}$

at which critical factors of length 3 occur.

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Here are the first two critical factors of t_{a} .





Lemma

The Thue-Morse word **m** (b = m = 2) is the unique word **t** containing a critical factor w of length $\ell > b$ such that $b \nmid \ell$. Moreover, $\ell = 3$.

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It had already been noticed that :

Proposition [Brlek, 1989]

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Square factors in m are of the form

(i) [\mu(0)]^2 or [\mu(1)]^2 or

(ii) [\mu(010)]^2 or [\mu(101)]^2,
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and have length either 2^{i+1} or $3 \cdot 2^{i+1}$.

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Corollary

If $b \leq m$, there exists a critical factor of **t** of length ℓ with $b \nmid \ell$ if and only if $gcd(b-1,m) \mid \ell$.

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Corollary

If $b \leq m$, there exists a critical factor of **t** of length ℓ with $b \nmid \ell$ if and only if $gcd(b-1,m) \mid \ell$.

This generalizes the following proposition :

Proposition [Allouche and Shallit, 2000]

t contains the square of a single letter if and only if gcd(b-1, m) = 1.