A note on the critical exponent of generalized Thue-Morse words

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The Thue-Morse word

The Thue-Morse word $\mathbf{m}$ is obtained as the fixpoint of the morphism

$$\begin{array}{rccc} \mu : & \Sigma^* & \rightarrow & \Sigma^* \\ & a & \mapsto & ab \\ & b & \mapsto & ba \end{array}$$

where $\Sigma = \{a, b\}$ is a binary alphabet, i.e.

$$\mu(\mathbf{m}) = \mathbf{m} = abbabaabbaababba \cdots$$

The Thue-Morse word can also be obtained from the numeric sequence

$$\Big(s_2(n) \bmod 2\Big)_{n\geq 0} = 01101001\cdots$$

where $s_2(n)$ denotes the sum of the digits of n written in base 2. Allouche and Shallit [2000] generalized this sequence :

$$\mathbf{t}_{b,m} = \Big(s_b(n) \bmod m\Big)_{n\geq 0}.$$

Rational power

The integer power of a word w is defined by

$$w^n = \begin{cases} \varepsilon & \text{if } n = 0 \\ ww^{n-1} & \text{if } n > 0 \end{cases}.$$

Definition

Let r be a rational such that $r|w| \in \mathbb{N}$. The rational power of a word w is defined by

$$w^r = w^{\lfloor r \rfloor} p,$$

where p is the prefix of w of length $(r - \lfloor r \rfloor)|w|$.

Rational power

Example

Consider the word *abb*. Then

$$
\begin{aligned}
(abb)^0 &= \varepsilon \\
(abb)^1 &= abb \\
(abb)^2 &= abbabb \\
(abb)^{2/3} &= ab \\
(abb)^{8/3} &= abbabbab \\
(abb)^{5/2} &\text{ is not defined}
\end{aligned}
$$

Rational power, index and critical exponent

$$x = baaaabbabbabbabaabbabbababababb$$

$$x = baaa\underbrace{abbabbabbab}_{(abb)^{11/3}}a\underbrace{abbabbab}_{(abb)^{8/3}}abababb$$

Rational powers of *abb* appear in x

Rational power, index and critical exponent

$$x = baaa\,\underbrace{abbabbabbab}_{(abb)^{11/3}}\,a\,\underbrace{abbabbab}_{(abb)^{8/3}}\,abababb$$

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Definition

The index of a word w in another word x is given by the number

$$\mathrm{Index}(w) = \sup\{r \in \mathbb{Q} \mid w^r \text{ is a factor of } x\}$$

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The index of abb in x is $11/3$.

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$$x = \overbrace{baaaa}^{a^4}\ldots$$

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$$x = baaaabbabbabbabaabbabb\overbrace{ababababab}^{(ba)^{9/2}}b$$

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$$\underbrace{abbabbabbab}_{(abb)^{11/3}} \qquad \underbrace{abbabbab}_{(abb)^{8/3}} \qquad \overbrace{babababab}^{(ba)^{9/2}}$$

Other powers appear in x.

Definition

The critical exponent $E(x)$ of a word x is the greatest index realized by its factor, i.e.

$$E(x) = \sup\{\text{Index}(w) \mid w \neq \varepsilon \text{ is a factor of } x\}.$$

Rational power, index and critical exponent

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The critical exponent of x is $9/2$.

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Some results on critical exponents

- Mignosi and Pirillo [1992] showed that the Fibonacci word $\mathbf{f}$ has critical exponent $E(\mathbf{f}) = 2 + \varphi$, where φ is the golden ratio.
- Dalia Krieger [2007] studied the critical exponent of fixpoint of non-erasing morphisms. In particular, she showed that for uniform morphisms, it is either infinite or rational.
- Allouche and Shallit [2000] showed that $E(\mathbf{t}_{b,m}) = 2$ for $b \leq m$.

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Our contribution is to compute $E(\mathbf{t}_{b,m})$ when $b > m$.

Generalized Thue-Morse word

Definition

Let $b \geq 2$, $m \geq 1$ be integers, Σ an alphabet of m letters, $\sigma : \Sigma \to \Sigma$ a cyclic permutation and $\mu : \Sigma^* \to \Sigma^*$ the morphism given by

$$\mu(\alpha) = \sigma^0(\alpha)\sigma^1(\alpha)\sigma^2(\alpha)\cdots\sigma^{b-1}(\alpha).$$

The generalized Thue-Morse word $\mathbf{t}$, beginning with $\bar{\alpha} \in \Sigma$, is the infinite word given by $\mathbf{t} = \mu^\omega(\bar{\alpha})$.

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Example

With $\Sigma = \{a, b\}$, $\sigma : a \mapsto b \mapsto a$ and $\bar{\alpha} = a$, we get $\mu(a) = ab$, $\mu(b) = ba$ and

$$\mathbf{t} = \mu^\omega(a) = abbabaabbaababba\cdots$$

which is the original Thue-Morse word $\mathbf{m}$.

The n-th letter

Remark

The n-th letter of $\mathbf{t}$ is $\sigma^{s_b(n)}(\bar{\alpha})$.

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Since 5 is written as 101 in base 2, the 5^{th} letter of the Thue-Morse word $\mathbf{m}$ is

$$\sigma^{s_2(5)}(a) = \sigma^{1+0+1}(a) = \sigma^2(a) = a,$$

as it should be :

$$\mathbf{m} = abbab \overset{5}{\overset{\uparrow}{a}} abbaababba \cdots \ .$$

Example : Recovering $\mathbf{t}_{b,m}$ from $\mathbf{t}$

Example

With

$$\Sigma = \mathbb{Z}_m$$

$$\sigma(i) = (i+1) \bmod m$$

$$\bar{\alpha} = 0,$$

the previous remark implies that the n-th letter of $\mathbf{t}$ is

$$\sigma^{s_b(n)}(0) = s_b(n) \bmod m,$$

which is the word $\mathbf{t}_{b,m}$ introduced by Allouche and Shallit [2000].

Example : $b = 5$, $m = 3$

Example

Let $b = 5$, $m = 3$, $\Sigma = \{\triangle, \diamondsuit, \heartsuit\}$ and $\sigma : \triangle \mapsto \diamondsuit \mapsto \heartsuit \mapsto \triangle$. This gives the following morphism

$$\begin{array}{llll} \mu & : & \triangle & \mapsto & \triangle\diamondsuit\heartsuit\triangle\diamondsuit \\ & & \diamondsuit & \mapsto & \diamondsuit\heartsuit\triangle\diamondsuit\heartsuit \\ & & \heartsuit & \mapsto & \heartsuit\triangle\diamondsuit\heartsuit\triangle \end{array}$$

By fixing $\bar{\alpha} = \diamondsuit$, we obtain the following generalized Thue-Morse word $\mathbf{t}_{\clubsuit} = \mu^{\omega}(\diamondsuit) = \diamondsuit$

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Example : $b = 7,\ m = 3$

Example

Let $b = 7,\ m = 3$, $\Sigma = \{0, 1, 2\}$ and $\sigma : 0 \mapsto 1 \mapsto 2 \mapsto 0$.
This gives the following morphism

$$\begin{array}{cccc} \mu & : & 0 & \mapsto & 0120120 \\ & & 1 & \mapsto & 1201201 \\ & & 2 & \mapsto & 2012012 \end{array}$$

and the following periodic fixpoint

$$\mathbf{t} = \mu^{\omega}(0) \quad = \quad 0$$

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Lemma [Morton and Mourant, 1991]

$\mathbf{t}$ is periodic if and only if $m \mid (b-1)$.

Theorem : the critical exponent

Assuming that $\mathbf{t}$ is aperiodic, i.e. $m \nmid b - 1$.

Theorem

The critical exponent of $\mathbf{t}$ *is given by*

$$E(\mathbf{t}) = \begin{cases} 2b/m & \text{if } b > m \\ 2 & \text{if } b \leq m \end{cases}$$

Theorem : the positions

We denote

$$S^p_{x,y} = \{s \in \mathbb{N}^+ \mid p = sx + ty, t \in \mathbb{Z}\}$$

the arithmetical progression associated to p, x and y.

Theorem : the positions

We denote

$$S_{x,y}^{p} = \{s \in \mathbb{N}^{+} \mid p = sx + ty, t \in \mathbb{Z}\}$$

the arithmetical progression associated to p, x and y.
Then, we consider the three following sets :

$$\begin{aligned} A &= \left\{ kb^q - b \mid k \in \mathbb{N}^{+}, b \nmid k \text{ and } q \in S_{b-1,m}^{m} \right\}, \\ B_N &= \left\{ kb^q - N \mid k \in \mathbb{N}^{+}, b \nmid k \text{ and } q \in S_{b-1,m}^{N} \right\}, \\ C &= (8 \cdot B_1 + 3) \cup (8 \cdot B_1 + 7). \end{aligned}$$

Theorem : the positions

Assuming that $\mathbf{t}$ is aperiodic, i.e. $m \nmid b - 1$.

Theorem

Critical factors of length Nb^i such that $b \nmid N$ occur in $\mathbf{t}$ at the following set of positions

$$\begin{cases} b^i A & \text{if } b > m, \\ b^i B_N & \text{if } b \leq m \text{ and } (b, m) \neq (2, 2), \\ b^i (B_1 \cup C) & \text{if } b = m = 2. \end{cases}$$

Example : $b = 5$ and $m = 3$

Example

Recall the generalized Thue-Morse word that has already been defined on $\Sigma = \{\triangle, \diamondsuit, \heartsuit\}$

$$\mathbf{t}_{\clubsuit} = \diamondsuit\heartsuit\triangle\diamondsuit\heartsuit\heartsuit\triangle\diamondsuit\heartsuit\triangle\triangle\diamondsuit\heartsuit\triangle\diamondsuit\diamondsuit\heartsuit\triangle\diamondsuit\heartsuit\cdots.$$

From the first Theorem, $E(\mathbf{t}_{\clubsuit}) = \frac{2b}{m} = \frac{10}{3}$.

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From the first Theorem, $E(\mathbf{t}_{\clubsuit}) = \frac{2b}{m} = \frac{10}{3}$.
From the second Theorem, we compute

$$S^3_{5-1,3} = \{3, 6, 9, 12, 15, \ldots\}$$

and obtain the set of positions

$$A = \{120, 245, 370, 495, 745, \ldots\}$$

at which critical factors of length 3 occur.

Example

Here are the first two critical factors of $\mathbf{t}_\clubsuit$.

$$\mathbf{t}_\clubsuit = \overset{0}{\diamondsuit}\overset{1}{\heartsuit}\overset{2}{\triangle}\overset{3}{\diamondsuit}\,\heartsuit\heartsuit\triangle\cdots\diamondsuit\heartsuit\triangle\,\underbrace{\overset{120}{\triangle}\,\diamondsuit\heartsuit\triangle\diamondsuit\heartsuit\triangle\diamondsuit\heartsuit\triangle}_{(\triangle\diamondsuit\heartsuit)^{10/3}}\triangle\diamondsuit\heartsuit\cdots$$

$$\cdots\heartsuit\triangle\diamondsuit\,\underbrace{\overset{245}{\diamondsuit}\,\heartsuit\triangle\diamondsuit\heartsuit\triangle\diamondsuit\heartsuit\triangle\diamondsuit}_{(\diamondsuit\heartsuit\triangle)^{10/3}}\diamondsuit\heartsuit\triangle\cdots$$

Lemma

The Thue-Morse word $\mathbf{m}$ $(b = m = 2)$ *is the unique word* $\mathbf{t}$ *containing a critical factor* w *of length* $\ell > b$ *such that* $b \nmid \ell$. *Moreover,* $\ell = 3$.

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The Thue-Morse word $\mathbf{m}$ *($b = m = 2$) is the unique word* $\mathbf{t}$ *containing a critical factor* w *of length* $\ell > b$ *such that* $b \nmid \ell$*. Moreover,* $\ell = 3$*.*

It had already been noticed that :

Proposition [Brlek, 1989]

Square factors in $\mathbf{m}$ are of the form

(i) $[\mu(0)]^2$ or $[\mu(1)]^2$ or

(ii) $[\mu(010)]^2$ or $[\mu(101)]^2$,

and have length either 2^{i+1} or $3 \cdot 2^{i+1}$.

More results

Corollary

If $b \leq m$, there exists a critical factor of $\mathbf{t}$ of length ℓ with $b \nmid \ell$ if and only if $\gcd(b-1, m) \mid \ell$.

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If $b \leq m$, there exists a critical factor of $\mathbf{t}$ of length ℓ with $b \nmid \ell$ if and only if $\gcd(b-1, m) \mid \ell$.

This generalizes the following proposition :

Proposition [Allouche and Shallit, 2000]

$\mathbf{t}$ contains the square of a single letter if and only if $\gcd(b-1, m) = 1$.